18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Topics in Algebraic Topology (18.917): Lecture 3

In this lecture we will establish some more of the basic properties of Steenrod operations. More precisely, we will show that the Steenrod squares are *stable* operations, and prove the Cartan formula which describes the interaction between Steenrod operations and multiplication in the cohomology of a space X. As before, we work in the setting of cochain complexes over the finite field $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ with two elements.

Let Ω denote the loop functor on complexes, so that we have canonical isomorphisms

$$(\Omega V)^n \simeq V^{n-1}$$

$$\mathrm{H}^n(\Omega V) \simeq \mathrm{H}^{n-1}(V).$$

Since the extended square functor $V \mapsto D_2(V)$ preserves acyclic objects, there is a canonical map

$$D_2(\Omega V) \xrightarrow{\phi} \Omega D_2(V)$$

for any complex V (see below for an explicit construction of this map).

The stability of the Steenrod operations is a consequence of the following result:

Proposition 1. *Let W be a complex and k an integer. Then the diagram*

$$\begin{array}{ccccc} \mathrm{H}^*(\Omega W) & & \xrightarrow{\sim} & & \mathrm{H}^{*-1}(W) \\ \downarrow{\scriptstyle \overline{\mathrm{Sq}}^k} & & & & \downarrow{\scriptstyle \overline{\mathrm{Sq}}^k} \\ \mathrm{H}^{*+k}(D_2(\Omega W)) & \longrightarrow & \mathrm{H}^{*+k}(\Omega(D_2 W)) & \xrightarrow{\sim} & \mathrm{H}^{*+k-1}(D_2(W)) \end{array}$$

is commutative.

Proof. Let $V = \Omega W$. Fix a class v in $\mathrm{H}^n(V)$, and let w denote the image of v in $\mathrm{H}^{n-1}(W)$. Without loss of generality, we may suppose that $V \simeq \mathbf{F}_2[-n]$ is generated by v, so that $W \simeq \mathbf{F}_2[1-n]$ is generated by w. We observe that $\mathrm{H}^{n+k-1}\, D_2(W)$ vanishes for $k \geq n$, so that the result is automatic. Let us therefore assume that $k < n$. In this case, $\mathrm{H}^{n+k-1}\, D_2 W$ and $\mathrm{H}^{n+k}\, D_2 V$ are 1-dimensional vector spaces, generated by $\overline{\mathrm{Sq}}^k(w)$ and $\overline{\mathrm{Sq}}^k(v)$, respectively. It will suffice to show that the map

$$\mathrm{H}^m\, D_2(V) \to \mathrm{H}^{m-1}\, D_2(W)$$

is an isomorphism for $m < 2n$.

Let U denote the complex

$$\ldots \to 0 \to \mathbf{F}_2 w \xrightarrow{\sim} \mathbf{F}_2 v \to 0 \to \ldots,$$

so we have a homotopy pullback diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & W. \end{array}$$

We obtain an associated diagram

$$\begin{array}{ccc} V^{\otimes 2} & \longrightarrow & U^{\otimes 2} \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & W^{\otimes 2}. \end{array}$$

The complex $\Omega W^{\otimes 2}$ can be identified with the kernel of the map f, which is given by the two term complex

$$\ldots \to 0 \to \mathbf{F}_2 v^2 \to \mathbf{F}_2 vw \oplus \mathbf{F}_2 wv \to 0 \to \ldots .$$

We therefore obtain a fiber sequence

$$V^{\otimes 2} \to \Omega W^{\otimes 2} \to \mathbf{F}_2^2[-2n+1]$$

of complexes with an action of the group Σ_2. The operation of taking homotopy coinvariants is exact, so we obtain a fiber sequence

$$D_2(V) \to \Omega D_2(W) \to \mathbf{F}_2[-2n+1].$$

The associated long exact sequence implies that $\mathrm{H}^m\, D_2(V) \simeq \mathrm{H}^{m-1}\, D_2(W)$ for $m < 2n$, as desired. □

To apply Proposition 1, we wish to study the relationship between symmetric multiplications and suspension. If V is a complex equipped with a symmetric multiplication $m : D_2(V) \to V$, then ΩV inherits a symmetric multiplication, given by the composition

$$D_2(\Omega V) \to \Omega D_2(V) \to \Omega V.$$

By construction, we have a commutative diagram

$$\begin{array}{ccc} H^{*+1} D_2(\Omega V) & \longrightarrow & H^{*+1}(\Omega V) \\ \downarrow \phi & & \downarrow \sim \\ H^* D_2(V) & \longrightarrow & H^* V \end{array}$$

where ϕ is the map appearing in Proposition 1. We immediately deduce the following:

Corollary 2. *Let V be a complex equipped with a symmetric multiplication. Then ΩV inherits a symmetric multiplication. Moreover, the canonical isomorphism*

$$\mathrm{H}^*\, V \simeq \mathrm{H}^{*+1}(\Omega V)$$

commutes with the Steenrod operations Sq^k.

Corollary 3. *Let X be a pointed topological space, and ΣX its suspension. Then the canonical isomorphism*

$$\mathrm{H}^*(X; \mathbf{F}_2) \simeq \mathrm{H}^{*+1}(\Sigma X; \mathbf{F}_2)$$

commutes with the action of the Steenrod operations Sq^k.

We can apply Corollary 3 to compute the Steenrod operations in some simple cases:

Example 4. Let $v \in \mathrm{H}^n_{\mathrm{red}}(S^n; \mathbf{F}_2)$ be the generator for the top cohomology of the n-sphere. Then

$$\mathrm{Sq}^k(v) = \begin{cases} v & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}.$$

To prove this, use Corollary 3 to reduce to the case $n = 0$. In this case, Example **??** shows that the operation Sq^0 is the identity on $\mathrm{H}^0_{\mathrm{red}}(S^0; \mathbf{F}_2)$.

Corollary 5. *Let X be a topological space, and let $v \in \mathrm{H}^n(X; \mathbf{F}_2)$. Then*

$$\mathrm{Sq}^k(x) = \begin{cases} x & \text{if } k = 0 \\ 0 & \text{if } k < 0. \end{cases}$$

Proof. Recall that the cohomology group $\mathrm{H}^n(X; \mathbf{F}_2)$ can be identified with the set of homotopy classes of maps from X into an Eilenberg-MacLane space $K(\mathbf{F}_2, n)$. More precisely, there exists a tautological cohomology class

$$\chi \in \mathrm{H}^n(K(\mathbf{F}_2, n); \mathbf{F}_2)$$

such that pulling back χ induces a bijection

$$\pi_0 \operatorname{Map}(X, K(\mathbf{F}_2, n)) \to \mathrm{H}^n(X; \mathbf{F}_2)$$

for every CW complex X. By general nonsense, we can reduce to the case $X = K(\mathbf{F}_2, n)$ and where $x = \chi$.

Let $v \in \mathrm{H}^n(S^n; \mathbf{F}_2)$ be the cohomology class described in Example 4. Then v induces a map

$$f : S^n \to K(\mathbf{F}_2, n).$$

The induced map

$$\mathrm{H}^{n+k}(K(\mathbf{F}_2, n); \mathbf{F}_2) \to \mathrm{H}^{n+k}(S^n; \mathbf{F}_2)$$

is injective (in fact, bijective) for $k \leq 0$. We may therefore reduce to the case where $X = S^n$ and $x = v$. The desired result now follows from Example 4. □

Warning 6. The negative Steenrod operations $\{\mathrm{Sq}^n\}_{n<0}$ act trivially on the cohomology of spaces, but are nontrivial in other examples. Similarly, Sq^0 acts by the identity on the cohomology of spaces, but not in general.

We now turn to the second main topic of this lecture: the Cartan formula. We begin by studying the interaction between the extended square functor D_2 and tensor products. Let V and W be complexes. We have equivalences

$$D_2(V) \otimes D_2(W) \simeq V^{\otimes 2}_{h\Sigma_2} \otimes W^{\otimes 2}_{h\Sigma_2} \simeq (V \otimes W)^{\otimes 2}_{h(\Sigma_2 \times \Sigma_2)}$$

$$D_2(V \otimes W) \simeq (V \otimes W)^{\otimes 2}_{h\Sigma_2}.$$

There is a canonical map

$$(V \otimes W)^{\otimes 2}_{h\Sigma_2} \to (V \otimes W)^{\otimes 2}_{h(\Sigma_2 \times \Sigma_2)},$$

given by the diagonal embedding of Σ_2 into $\Sigma_2 \times \Sigma_2$. This induces a map $\psi : D_2(V \otimes W) \to D_2(V) \otimes D_2(W)$.

Proposition 7. *Let V and W be complexes. Let $v \in \mathrm{H}^m V$, $w \in \mathrm{H}^n W$, so that we can form a class $v \otimes w \in \mathrm{H}^{m+n}(V \otimes W)$. For every integer k, we have an equality*

$$\psi\, \overline{\mathrm{Sq}}^k(v \otimes w) = \Sigma_{k=k'+k''}\, \overline{\mathrm{Sq}}^{k'}(v) \otimes \overline{\mathrm{Sq}}^{k''}(w)$$

in the cohomology group $\mathrm{H}^{m+n+k}(D_2(V) \otimes D_2(W))$.

Remark 8. The sum in this expression is well-defined, since $\overline{\mathrm{Sq}}^{k'}(v) \otimes \overline{\mathrm{Sq}}^{k''}(w)$ vanishes for $k' > m$ or $k'' > n$. There are only finitely many terms which do not satisfy either condition.

Proof. If $k > m + n$, then the result is obvious since both sides vanish. Let us therefore assume that $k = m + n - i$, where $i \geq 0$. We can rewrite the equation

$$\psi\, \overline{\mathrm{Sq}}^{m+n-i}(v \otimes w) = \Sigma_{i=i'+i''}\, \overline{\mathrm{Sq}}^{m-i'}(v) \otimes \overline{\mathrm{Sq}}^{n-i''}(w),$$

where the sum is taken over $i', i'' \geq 0$.

Without loss of generality, we may assume that $V = \mathbf{F}_2[-m]$ and $W = \mathbf{F}_2[-n]$. In this case, we have canonical isomorphisms

$$\mathrm{H}^*(D_2(V)) \simeq \mathrm{H}_{2m-*}(B\Sigma_2; \mathbf{F}_2)e_{2m}$$

$$\mathrm{H}^*(D_2(W)) \simeq \mathrm{H}_{2n-*}(B\Sigma_2; \mathbf{F}_2)e_{2n}.$$

$$\mathrm{H}^*(D_2(V \otimes W)) \simeq \mathrm{H}_{2m+2n-*}(B\Sigma_2; \mathbf{F}_2)e_{2m+2n}.$$

For each $j \geq 0$, let x_j denote a generator of $\mathrm{H}_j(B\Sigma_2; \mathbf{F}_2)$. Under the identifications above, we have

$$\overline{\mathrm{Sq}}^{m+n-i}(v \otimes w) \mapsto x_i e_{2m+2n}$$

$$\overline{\mathrm{Sq}}^{m-i'}(v) \mapsto x_{i'} e_{2m}$$

$$\overline{\mathrm{Sq}}^{n-i''}(w) \mapsto x_{i''} e_{2n}.$$

Moreover, the map ψ simply corresponds to the comultiplication

$$\Psi : \mathrm{H}_*(B\Sigma_2; \mathbf{F}_2) \to \mathrm{H}_*(B\Sigma_2; \mathbf{F}_2) \otimes \mathrm{H}_*(B\Sigma_2; \mathbf{F}_2)$$

on the homology of the space $B\Sigma_2$. The cohomology ring $\mathrm{H}^*(B\Sigma_2; \mathbf{F}_2) \simeq \mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2)$ is simply isomorphic to a polynomial ring $\mathbf{F}_2[t]$ having a basis $\{t^j\}_{j \geq 0}$. The corresponding comultiplication is given in the dual basis $\{x_i\}_{i \geq 0}$ by the formula

$$x_i \mapsto \sum_{i'+i''} x_{i'} \otimes x_{i''}.$$

We now simply compute

$$\overline{\mathrm{Sq}}^{m+n-i}(v \otimes w) = x_i e_{2m+2n} \mapsto \sum_{i=i'+i''} (x_{i'} e_{2m}) \otimes (x_{i''} e_{2n}) = \overline{\mathrm{Sq}}^{m-i'}(v) \otimes \overline{\mathrm{Sq}}^{n-i''}(w)$$

to obtain the desired formula. □

For any complex V equipped with a symmetric multiplication $m : D_2(V) \to V$, we can form a diagram

$$\begin{array}{ccccccc}
D_2(V \otimes V) & \longrightarrow & D_2(D_2(V)) & \xrightarrow{D_2(m)} & D_2(V) & & \\
\downarrow & & & & & \searrow^{m} & \\
D_2(V) \otimes D_2(V) & \xrightarrow{m \otimes m} & V \otimes V & \longrightarrow & D_2(V) & \xrightarrow{m} & V.
\end{array}$$

If m is good (see Lecture 4), then this diagram commutes up to homotopy. Passing to cohomology and applying Proposition 7, we deduce the following:

Corollary 9. *Let V be a complex equipped with a good symmetric multiplication. Then, for every pair of elements $v, w \in \mathrm{H}^*(V)$, the Cartan formula holds:*

$$\mathrm{Sq}^k(vw) = \sum_{k=k'+k''} \mathrm{Sq}^{k'}(v)\,\mathrm{Sq}^{k''}(w).$$

Corollary 10. *Let X be a topological space, and let $x, y \in \mathrm{H}^*(X; \mathbf{F}_2)$. Then, for each $n \geq 0$,*

$$\mathrm{Sq}^n(xy) = \sum_{n=n'+n''} \mathrm{Sq}^{n'}(x)\,\mathrm{Sq}^{n''}(y).$$

It is convenient to summarize Corollary 10 by asserting that the *total Steenrod square* $x \mapsto \sum_{n \geq 0} \mathrm{Sq}^n(x)$ is a multiplicative operation.

We can now compute the action of the Steenrod algebra in a situation where they are definitely nontrivial:

Corollary 11. *Let* $\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2) = \mathbf{F}_2[t]$. *Then the action of the Steenrod algebra on* $\mathbf{F}_2[t]$ *can be described by the following formula:*

$$\mathrm{Sq}^k\, t^n = \binom{n}{k} t^{n+k}.$$

Here $\binom{n}{k}$ *denotes the binomial coefficient*

$$\frac{n!}{k!(n-k)!}$$

if $0 \leq k \leq n$; *by convention we will agree that* $\binom{n}{k}$ *vanishes otherwise.*

Proof. Let Sq denote the operation $x \mapsto \sum_{n \geq 0} \mathrm{Sq}^n(x)$. Since t has degree 1, $\mathrm{Sq}^n(t)$ vanishes for $n > 1$ and is equal to t^2 when $t = 1$. It follows that $\mathrm{Sq}(t) = \mathrm{Sq}^0(t) + \mathrm{Sq}^1(t) = t + t^2$. Since the operation Sq is multiplicative, we have

$$\mathrm{Sq}(t^n) = (t + t^2)^n = \sum_{0 \leq k \leq n} \binom{n}{k} t^{n+k}.$$

The desired result now follows by extracting individual coefficients. □

Warning 12. Our convention that $\binom{n}{k}$ vanishes for $n < 0$ is somewhat nonstandard. For example, it has the consequence that $\binom{n}{k}$ is *not* a polynomial function of n, even for $k = 1$.

The cohomology ring $\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2)$ is a very important example which will play a large role in the later part of this course.