18.906: Problem Set VI

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet.

Extra credit for finding mistakes and telling me about them early!

28. **(a)** Suppose that X is simply connected space. Show that $H_*(X)$ is of finite type (finitely generated as abelian group in each dimension) if and only if $H_*(\Omega X)$ is. Similarly, show that $\overline{H}_*(X)$ is entirely p-torsion if and only if $\overline{H}_*(\Omega X)$ is entirely p-torsion.

(b) Show that if A is a finitely generated abelian group then $H_*(K(A,n))$ is of finite type for any $n > 1$. (We did the case $n = 1$ in class.) Show that if A is p-torsion then the same is true for $\overline{H}_i(K(A,n))$ for any n and i.

29. We found that $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}$. Explain why this shows that $\pi_4(S^2) = \mathbb{Z}/2\mathbb{Z}$ as well, and describe a non-null map $S^4 \to S^2$.

30. An *immersion* of manifold M into manifold N is a smooth map $f : M \to N$ that induces an injection on tangent bundles. Draw some immersions of the circle in $\mathbb{R}^2$. The "Whitney-Graustein theorem" classifies them. Find out about the Boys surface, an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$. Whitney proved that any closed n-manifold immerses in codimension $n - 1$.

An immersion $f : M \to N$ determines an embedding $\tau_M \hookrightarrow f^*\tau_N$ of vector bundles over M. In particular, an immersion $M \to \mathbb{R}^n$ determines an embedding $\tau_M \hookrightarrow n\epsilon$. Not every m-plane bundle admits an $(n-m)$-dimensional complement; this provides obstructions to possible immersions. In fact Smale-Hirsch theory shows that this is the only obstruction; any map $f : M \to N$ that is covered by a fiberwise linear embedding of the tangent bundles (not necessarily given by the derivative of the map!) f is homotopic to an immersion.

Use this idea, together with Stiefel-Whitney classes, to put a lower bound on the codimension of an immersion of $\mathbb{R}P^n$ into Euclidean space. (Determination the minimal codimension of an immersion of $\mathbb{R}P^n$ in general is a very difficult computational problem, still not completely resolved.)

Using these examples, find, for each n, a closed n-manifold that does not immerse into $\mathbb{R}^{2n-\alpha(n)-1}$, where $\alpha(n)$ is the sum of the digits in the binary expansion of n.

This is a best possible result, according to a theorem of Ralph Cohen (following work of Ed Brown and Frank Peterson): every closed n-manifold immerses in codimension $n - \alpha(n)$.

31. **(a)** Let ξ be a complex n-plane bundle and λ a complex line bundle, both over a space B. Find an explicit expression for the Chern classes of $\lambda \otimes \xi$ in terms of the

Chern classes of ξ and the Euler class of λ.

(b) Let ξ be an real n-plane bundle over B. Show that if n is odd then $2e(\xi) = 0$ in $H^n(B)$. Give an example in which $e(\xi)$ is nevertheless nonzero.

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