25 Künneth and Eilenberg-Zilber

We want to compute the homology of a product. Long ago, in Lecture 7, we constructed a bilinear map $S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, called the cross product. So we get a linear map $S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y)$, and it satisfies the Leibniz formula, i.e., $d(x \times y) = dx \times y + (-1)^p x \times dy$. The method we used works with any coefficient ring, not just the integers.

Definition 25.1. Let C_*, D_* be two chain complexes. Their *tensor product* is the chain complex with

$$(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q \,.$$

The differential $(C_* \otimes D_*)_n \to (C_* \otimes D_*)_{n-1}$ sends $C_p \otimes D_q$ into the submodule $C_{p-1} \otimes D_q \bigoplus C_p \otimes D_{q-1}$ by

$$x \otimes y \mapsto dx \otimes y + (-1)^p x \otimes dy \,.$$

So the cross product is a map of chain complexes $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$. There are two questions:
(1) Is this map an isomorphism in homology?
(2) How is the homology of a tensor product of chain complexes related to the tensor product of their homologies?

It's easy to see what happens in dimension zero, because $\pi_0(X) \times \pi_0(Y) = \pi_0(X \times Y)$ implies that $H_0(X) \otimes H_0(Y) \xrightarrow{\cong} H_0(X \times Y)$.

Let's dispose of the purely algebraic question (2) first.

Theorem 25.2. *Let R be a PID and C_*, D_* be chain complexes of R-modules. Assume that C_n is a free R-module for all n. There is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C), H_q(D)) \to 0$$

natural in these data, that splits (but not naturally).

Proof. This is exactly the same as the proof for the UCT. It's a good idea to work through this on your own. □

Corollary 25.3. *Let R be a PID and assume C'_n and C_n are R free for all n. If $C'_* \to C_*$ and $D'_* \to D_*$ are homology isomorphisms then so is $C'_* \otimes D'_* \to C_* \otimes D_*$.*

Our attack on question (1) is via the method of "acyclic models." This is really a special case of the Fundamental Theorem of Homological Algebra, Theorem 22.1.

Definition 25.4. Let $\mathcal{C}$ be a category, and fix a set $\mathcal{M}$ of objects in $\mathcal{C}$, to be called the "models." A functor $F : \mathcal{C} \to \mathbf{Ab}$ is *$\mathcal{M}$-free* if it is the free abelian group generated by a coproduct of corepresentable functors. That is, F is a direct sum of functors of the form $\mathbf{Z}\mathcal{C}(M, -)$ where $M \in \mathcal{M}$.

Example 25.5. Since we are interested in the singular homology of a product of two spaces, it may be sensible to take as $\mathcal{C}$ the category of ordered pairs of spaces, $\mathcal{C} = \mathbf{Top}^2$, and for $\mathcal{M}$ the set of pairs of simplicies, $\mathcal{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}$. Then

$$S_n(X \times Y) = \mathbf{Z}[\mathbf{Top}(\Delta^n \times X) \times \mathbf{Top}(\Delta^n, Y)] = \mathbf{Z}\mathbf{Top}^2((\Delta^n, \Delta^n), (X, Y)) \,.$$

is $\mathcal{M}$-free.

Example 25.6. With the same category and models,

$$(S_*(X) \otimes S_*(Y))_n = \bigoplus_{p+q=n} S_p(X) \otimes S_q(Y) \,,$$

is $\mathcal{M}$-free, since the tensor product has as free basis the set

$$\coprod_{p+q=n} \mathrm{Sin}_p(X) \times \mathrm{Sin}_q(Y) = \coprod_{p+q=n} \mathbf{Top}^2((\Delta^p, \Delta^q), (X, Y))\,.$$

Definition 25.7. A natural transformation of functors $\theta : F \to G$ is an $\mathcal{M}$-*epimorphism* if $\theta_M : F(M) \to G(M)$ is a surjection of abelian groups for every $M \in \mathcal{M}$. A *sequence* of natural transformations is a composable pair $G' \to G \to G''$ with trivial composition. Let K be the objectwise kernel of $G \to G''$. There is a factorization $G' \to K$. The sequence is $\mathcal{M}$-*exact* if $G' \to K$ is a $\mathcal{M}$-epimorphism. Equivalently, $G'(M) \to G(M) \to G''(M)$ is exact for all $M \in \mathcal{M}$.

Example 25.8. We claim that

$$\cdots \to S_n(X \times Y) \to S_{n-1}(X \times Y) \to \cdots \to S_0(X \times Y) \to H_0(X \times Y) \to 0$$

is $\mathcal{M}$-exact. Just plug in (Δ^p, Δ^q): you get an exact sequence, since $\Delta^p \times \Delta^q$ is contractible.

Example 25.9. The sequence

$$\cdots \to (S_*(X) \otimes S_*(Y))_n \to (S_*(X) \otimes S_*(Y))_{n-1} \to \cdots \to S_0(X) \otimes S_0(Y) \to H_0(X) \otimes H_0(Y) \to 0\,.$$

is also $\mathcal{M}$-exact, by Corollary 25.3.

The terms "$\mathcal{M}$-free" and "$\mathcal{M}$-exact" relate to each other in the expected way:

Lemma 25.10. *Let $\mathcal{C}$ be a category with a set of models $\mathcal{M}$ and let $F, G, G' : \mathcal{C} \to \mathbf{Ab}$ be functors. Suppose that F is $\mathcal{M}$-free, let $G' \to G$ be a $\mathcal{M}$-epimorphism, and let $f : F \to G$ be any natural transformation. Then there is a lifting:*

$$\begin{tikzcd} & G' \arrow[d] \\ F \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"'] & G \end{tikzcd}$$

Proof. Clearly we may assume that $F(X) = \mathbf{Z}\mathcal{C}(M, X)$. Suppose that $X = M \in \mathcal{M}$. We get:

$$\begin{tikzcd} & G'(M) \arrow[d] \\ \mathbf{Z}\mathcal{C}(M, M) \arrow[ur, dashed, "\overline{f}_M"] \arrow[r, "f_M"'] & G(M) \end{tikzcd}$$

Consider $1_M \in \mathbf{Z}\mathcal{C}(M, M)$. Its image $f_M(1_M) \in G(M)$ is hit by some element in $c_M \in G'(M)$, since $G' \to G$ is an $\mathcal{M}$-epimorphism. Define $\overline{f}_M(1_M) = c_M$.

Now we exploit naturality! Any $\varphi : M \to X$ produces a commutative diagram

$$\begin{tikzcd} \mathcal{C}(M, M) \arrow[r, "\overline{f}_M"] \arrow[d, "\varphi_*"] & G'(M) \arrow[d, "\varphi_*"] \\ \mathcal{C}(M, X) \arrow[r, "\overline{f}_X"] & G'(X) \end{tikzcd}$$

Chase 1_M around the diagram, to see what the value of $\overline{f}_X(\varphi)$ must be:

$$\overline{f}_X(\varphi) = \overline{f}_X(\varphi_*(1_M)) = \varphi_*(\overline{f}_M(1_M)) = \varphi_*(c_M)\,.$$

Now extend linearly. You should check that this does define a natural transformation. $\square$

This is precisely the condition required to prove the Fundamental Theorem of Homological Algebra. So we have the

Theorem 25.11 (Acyclic Models)**.** *Let $\mathcal{M}$ be a set of models in a category $\mathcal{C}$. Let $\theta : F \to G$ be a natural transformation of functors from $\mathcal{C}$ to* **Ab**. *Let F_* and G_* be functors from $\mathcal{C}$ to chain complexes, with augmentations $F_0 \to F$ and $G_0 \to G$. Assume that F_n is $\mathcal{M}$-free for all n, and that $G_* \to G \to 0$ is an $\mathcal{M}$-exact sequence. Then there is a unique chain homotopy class of chain maps $F_* \to G_*$ covering θ.*

Corollary 25.12. *Suppose furthermore that θ is a natural isomorphism. If each G_n is $\mathcal{M}$-free and $F_* \to F \to 0$ is an $\mathcal{M}$-exact sequence, then any natural chain map $F_* \to G_*$ covering θ is a natural chain homotopy equivalence.*

Applying this to our category $\mathbf{Top}^2$ with models as before, we get the following theorem that completes work we did in Lecture 7.

Theorem 25.13 (Eilenberg-Zilber theorem)**.** *There are unique chain homotopy classes of natural chain maps:*

$$S_*(X) \otimes S_*(Y) \leftrightarrows S_*(X \times Y)$$

covering the usual isomorphism

$$H_0(X) \otimes H_0(Y) \cong H_0(X \times Y)\,,$$

and they are natural chain homotopy inverses.

Corollary 25.14. *There is a canonical natural isomorphism $H(S_*(X) \otimes S_*(Y)) \cong H_*(X \times Y)$.*

Combining this theorem with the algebraic Künneth theorem, we get:

Theorem 25.15 (Künneth theorem)**.** *Take coefficients in a PID R. There is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes_R H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y)) \to 0$$

natural in X, Y. It splits as R-modules, but not naturally.

Example 25.16. If $R = k$ is a field, every module is free, so the Tor term vanishes, and you get a Künneth *isomorphism*:

$$\times : H_*(X; k) \otimes_k H_*(Y; k) \xrightarrow{\cong} H_*(X \times Y; k)$$

This is rather spectacular. For example, what is $H_*(\mathbf{RP}^3 \times \mathbf{RP}^3; k)$, where k is a field? Well, if k has characteristic different from 2, $\mathbf{RP}^3$ has the same homology as S^3, so the product has the same homology as $S^3 \times S^3$: the dimensions are $1, 0, 0, 2, 0, 0, 1$. If $\operatorname{char} k = 2$, on the other hand, the cohomology modules are either 0 or k, and we need to form the graded tensor product:

$$\begin{matrix} k & k & k & k \\ k & k & k & k \\ k & k & k & k \\ k & k & k & k \end{matrix}$$

so the dimensions of the homology of the product are $1, 2, 3, 4, 3, 2, 1$.

The palindromic character of this sequence will be explained by Poincaré duality. Let's look also at what happens over the integers. Then we have the table of tensor products

	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$
$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$
$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}/2\mathbf{Z}$
0	0	0	0	0
$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$

There is only one nonzero Tor group, namely

$$\mathrm{Tor}_1^{\mathbf{Z}}(H_1(\mathbf{RP}^3), H_1(\mathbf{RP}^3)) = \mathbf{Z}/2\mathbf{Z}.$$

Putting this together, we get the groups

H_0	$\mathbf{Z}$
H_1	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
H_2	$\mathbf{Z}/2\mathbf{Z}$
H_3	$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
H_4	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
H_5	0
H_6	$\mathbf{Z}$

The failure of perfect symmetry here is interesting, and will also be explained by Poincaré duality.