20 Tensor product

The category of R-modules is what might be called a "categorical ring," in which addition corresponds to the direct sum, the zero element is the zero module, 1 is R itself, and multiplication is . . . well, the subject for today. We care about the tensor product for two reasons: First, it allows us to deal smoothly with bilinear maps such that the cross-product. Second, and perhaps more important, it will allow us relate homology with coefficients in an any R-module to homology with coefficients in the PID R; for example, relate $H_*(X; M)$ to $H_*(X)$, where M is any abelian group.

Let's begin by recalling the definition of a bilinear map over a commutative ring R.

Definition 20.1. Given three R-modules, M, N, P, a *bilinear map* (or, to be explicit, *R-bilinear map*) is a function $\beta : M \times N \to P$ such that

$$\beta(x + x', y) = \beta(x, y) + \beta(x', y)\,, \quad \beta(x, y + y') = \beta(x, y) + \beta(x, y')\,,$$

and

$$\beta(rx, y) = r\beta(x, y)\,, \quad \beta(x, ry) = r\beta(x, y)\,,$$

for $x, x' \in M$, $y, y' \in N$, and $r \in R$.

Example 20.2. $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by the dot product is an $\mathbf{R}$-bilinear map. The cross product $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is $\mathbf{R}$-bilinear. If R is a ring, the multiplication $R \times R \to R$ is R-bilinear, and the multiplication on an R-module M given by $R \times M \to M$ is R-bilinear. This enters into topology because the cross-product $H_m(X;R) \times H_n(Y;R) \xrightarrow{\times} H_{m+n}(X \times Y;R)$ is R-bilinear.

Wouldn't it be great to reduce stuff about bilinear maps to linear maps? We're going to do this by means of a universal property.

Definition 20.3. Let M, N be R-modules. A *tensor product* of M and N is an R-module P and a bilinear map $\beta_0 : M \times N \to P$ such that for every R-bilinear map $\beta : M \times N \to Q$ there is a unique factorization

$$\begin{array}{ccc} M \times N & \xrightarrow{\beta_0} & P \\ & \searrow^{\beta} & \downarrow{\scriptstyle f} \\ & & Q \end{array}$$

through an R-module homomorphism f.

We should have pointed out that the composition $f \circ \beta_0$ is indeed again R-bilinear; but this is easy to check.

So β_0 is a universal bilinear map out of $M \times N$. Instead of β_0 we're going to write $\otimes : M \times N \to P$. This means that $\beta(x,y) = f(x \otimes y)$ in the above diagram. There are lots of things to say about this. When you have something that is defined via a universal property, you know that it's unique . . . but you still have to check that it exists!

Construction 20.4. I want to construct a univeral R-bilinear map out of $M \times N$. Let $\beta : M \times N \to Q$ be any R-bilinear map. This β isn't linear. Maybe we should first extend it to a linear map. There is a unique R-linear extension over the free R-module $R\langle M \times N\rangle$ generated by the set $M \times N$:

$$\begin{array}{ccccc} M \times N & & \xrightarrow{\quad\beta\quad} & & Q \\ & \searrow^{[-]} & & \nearrow_{\overline{\beta}} & \\ & & R\langle M \times N\rangle & & \end{array}$$

The map $[-]$, including a basis, isn't bilinear. So we should quotient $R\langle M \times N\rangle$ by a submodule S of relations to make it bilinear. So S is the sub R-module generated by the four familes of elements (corresponding to the four relations in the definition of R-bilinearity):

1. $[(x+x', y)] - [(x, y)] - [(x' - y)]$

2. $[(x, y+y')] - [(x, y)] - [(x, y')]$

3. $[(rx, y)] - r[(x, y)]$

4. $[(x, ry)] - r[(x, y)]$

for $x, x' \in M$, $y, y' \in N$, and $r \in R$. Now the composite $M \times N \to R\langle M \times N\rangle/S$ *is* R-bilinear - we've quotiented out by all things that prevented it from being so! And the map $R\langle M \times N\rangle \to Q$ factors as $R\langle M \times N\rangle \to R\langle M \times N\rangle/S \xrightarrow{f} Q$, where f is R-linear, and uniquely because the map to the quotient is surjective. This completes the construction.

If you find yourself using this construction, stop and think about what you're doing. You're never going to use this construction to compute anything. Here's an example: for any abelian group A,

$$A \times \mathbf{Z}/n\mathbf{Z} \to A/nA\,, \quad (a, b) \mapsto ba \mod nA$$

is clearly bilinear, and is universal as such. Just look: If $\beta : A \times \mathbf{Z}/n\mathbf{Z} \to Q$ is bilinear then $\beta(na, b) = n\beta(a, b) = \beta(a, nb) = \beta(a, 0) = 0$, so β factors through A/nA; and $A \times \mathbf{Z}/n\mathbf{Z} \to A/nA$ is surjective. So $A \otimes \mathbf{Z}/n\mathbf{Z} = A/nA$.

Remark 20.5. The image of $M \times N$ in $R\langle M \times N\rangle/S$ generates it as an R-module. These elements $x \otimes y$ are called "decomposable tensors."

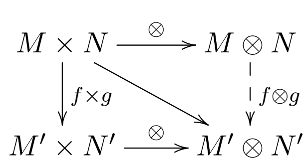
What are the properties of such a universal bilinear map?

Property 20.6 (Uniqueness)**.** Suppose $\beta_0 : M \times N \to P$ and $\beta_0' : M \times N \to P'$ are both universal. Then there's a linear map $f : P \to P'$ such that $\beta_0' = f\beta_0$ and a linear map $f' : P' \to P$ such that $\beta_0 = f'\beta_0'$. The composite $f'f : P \to P$ is a linear map such that $f'f\beta_0 = f'\beta_0' = \beta_0$. The identity map is another. But by universality, there's only one such linear map, so $f'f = 1_P$. An identical argument shows that $ff' = 1_{P'}$ as well, so they are inverse linear isomorphism. In brief:

> The target of a univeral R-bilinear map $\beta_0 : M \times N \to P$ is unique up to a unique R-linear isomorphism compatible with the map β_0.

This entitles us to speak of "the" universal bilinear map out of $M \times N$, and give the target a symbol: $M \otimes_R N$. If R is the ring of integers, or otherwise understood, we will drop it from the notation.

Property 20.7 (Functoriality)**.** Suppose $f : M \to M'$ and $: N \to N'$. Study the diagram



There is a unique R-linear map $f \otimes g$ because the diagonal map is R-bilinear and the map $M \times N \to M \otimes N$ is the universal R-bilinear map out of $M \times N$. You are invited to show that this construction is functorial.

Property 20.8 (Unitality, associativity, commutativity)**.** I said that this was going to be a "categorical ring," so we should check various properties of the tensor product. For example, $R \otimes_R M$ should be isomorphic to M. Let's think about this for a minute. We have an R-bilinear map $R \times M \to M$, given by multiplication. We just need to check the universal property. Suppose we have an R-bilinear map $\beta : R \times M \to P$. We have to construct a map $f : M \to P$ such that $\beta(r, x) = f(rx)$ and show it's unique. Our only choice is $f(x) = \beta(1, x)$, and that works.

Similarly, we should check that there's a unique isomorphism $L \otimes (M \otimes N) \xrightarrow{\cong} (L \otimes M) \otimes N$ that's compatible with $L \times (M \times N) \cong (L \times M) \times N$, and that there's a unique isomorphism $M \otimes N \to N \otimes M$ that's compatible with the switch map $M \times N \to N \times M$. There are a few other things to check, too: Have fun!

Property 20.9 (Sums)**.** What happens with $M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$? This might be a finite direct sum, or maybe an uncountable collection. How does this relate to $\bigoplus_{\alpha \in A}(M \otimes N_\alpha)$? Let's construct a map

$$f : \bigoplus_{\alpha \in A} (M \otimes N_\alpha) \to M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha \right) .$$

We just need to define maps $M \otimes N_\alpha \to M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$ because the direct sum is the coproduct. We can use $1 \otimes \mathrm{in}_\alpha$ where $\mathrm{in}_\alpha : N_\alpha \to \bigoplus_{\alpha \in A} N_\alpha$. These give you a map f.

What about a map the other way? We'll define a map out of the tensor product using the universal property. So we need to define a bilinear map out of $M \times \left(\bigoplus_{\alpha \in A} N_\alpha\right)$. By linearity in the second factor, it will suffice to say where to send elements of the form $(x, y) \in M \otimes N_\beta$. Just send it to $x \otimes \mathrm{in}_\beta y$, where $\mathrm{in}_\beta : N_\beta \to \bigoplus_{\alpha \in A} N_\alpha$ is the inclusion of a summand. It's up to you to check that these are inverses.

Property 20.10 (Distributivity)**.** Suppose $f : M' \to M$, $r \in R$, and $g_0, g_1 : N' \to N$. Then

$$f \otimes (g_0 + g_1) = f \otimes g_0 + f \otimes g_1 : M' \otimes N' \to M \otimes N$$

and

$$f \otimes r g_0 = r(f \otimes g_0) : M' \otimes N' \to M \otimes N .$$

Again I'll leave this to you to check.

Our immediate use of this construction is to give a clean definition of "homology with coefficients in M," where M is any abelian group. First, endow singular chains with coefficients in M like this:

$$S_*(X; M) = S_*(X) \otimes M$$

Then we define

$$H_n(X; M) = H_n(S_*(X; M)) .$$

Since $S_n(X) = \mathbf{Z}\mathrm{Sin}_n(X)$, $S_n(X; M)$ is a direct sum of copies of M indexed by the n-simplices in X. If M happens to be a ring, this coincides with the notation used in the last lecture. The boundary maps are just $d \otimes 1 : S_n(X) \otimes M \to S_{n-1}(X) \otimes M$.

As we have noted, the sequence

$$0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$$

is split short exact, and therefore applying the functor $- \otimes M$ to it produces another split short exact sequence. So

$$S_n(X, A) \otimes M = S_n(A; M)/S_n(X; M) ,$$

and it makes sense to use the notation $S_n(X, A; M)$ for this. This is again a chain complex (by functoriality of the tensor product), and we define

$$H_n(X, A; M) = H_n(S_n(X, A; M)) .$$

Notice that

$$H_n(*; M) = \begin{cases} M & \text{for } n = 0 \\ 0 & \text{otherwise} . \end{cases}$$

The following result is immediate:

Proposition 20.11. *For any abelian group M, $(X, A) \mapsto H_*(X, A; M)$ provides a homology theory satisfying the Eilenberg-Steenrod axioms with $H_0(*; M) = M$.*

Suppose R is a commutative ring and A is an abelian group. Then $A \otimes R$ is naturally an R-module. So $S_*(X; R)$ is a chain complex of R-modules – *free* R-modules. We can go a little further: suppose that M is an R-module. Then $A \otimes M$ is an R-module; and $S_*(X; M)$ is a chain complex of R-modules. We can also write

$$S_*(X; M) = S_*(X; R) \otimes_R M .$$

This construction is natural in the R-module M; and, again using the fact that sums of exact sequences are exact, a short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0$$

leads to a short exact sequence of chain complexes

$$0 \to S_*(X; M') \to S_*(X; M) \to S_*(X; M'') \to 0$$

and hence to a long exact sequence in homology, a "coefficient long exact sequence":

$$\cdots \longrightarrow H_{n+1}(X; M'') \xrightarrow{\partial} H_n(X; M') \longrightarrow H_n(X; M) \longrightarrow H_n(X; M'') \xrightarrow{\partial} H_{n-1}(X; M') \longrightarrow \cdots .$$

A particularly important case is when R is a field; then $S_*(X; R)$ is a chain complex of vector spaces over R, and $H_*(X; R)$ is a graded vector space over R.

Question 20.12. A reasonable question is this: Suppose we know $H_*(X)$. Can we compute $H_*(X; M)$ for an abelian group M? More generally, suppose we know $H_*(X; R)$ and M is an R-module. Can we compute $H_*(X; M)$?