

An introduction to simplicial sets

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1 Introduction

This is an elementary introduction to simplicial sets, which are generalizations of Δ -complexes from algebraic topology. The theory of simplicial sets provides a way to express homotopy and homology without requiring topology. This paper is meant to be accessible to anyone who has had experience with algebraic topology and has at least basic knowledge of category theory.

An important part of simplicial homology is the idea of using Δ -complexes instead of simplicial complexes (see [6, Ch. 2]). They allow one to deal with the combinatorial data associated with a simplicial complex (which is important for homology) instead of the actual topological structure (which is not).

Another simplex-based homology theory is singular homology, whose singular maps (see [6, Ch. 2]) represent simplices in a given topological space. While singular maps have properties analogous to simplices, such as a sensible definition for the faces of a singular map, singular maps are not in general injective, which means the data for gluing the faces of a singular map together might not be able to be described as a Δ -complex.

The theory of simplicial sets generalizes the idea of Δ -complexes to encompass other objects with simplex structure, such as singular maps. The theory provides a realization functor $|-|$ from simplicial sets to topological spaces which preserves homotopy. This functor is left adjoint to the functor S which takes topological spaces and gives a simplicial set consisting of the singular maps.

Perhaps the most difficult part for a newcomer to the subject of simplicial sets is getting used to the category theory involved. Because of this, this paper limits discussion to simplicial sets and algebraic topology.

The author found [1] very useful when trying to understand the idea of simplicial sets and [4] illuminating for the derivation of the relations (8), (9), and (10). Much of the material comes from [3], but it was corroborated with [2] to determine what is modern notation.

2 Simplicial sets

In this section, we define simplicial sets without providing motivation, and we describe the combinatorial data necessary for specifying a simplicial set. We then try to build intuition by bringing in the geometric notion of simplices from algebraic topology.

We first define the category $\mathbf{\Delta}$, a visualization of which is Example 3.

Definition 1. Let $\mathbf{\Delta}$ be the category whose objects are finite sets $\{0, 1, 2, \dots, n\}$ and whose morphisms are order-preserving functions (i.e., functions $f : \{0, \dots, n\} \rightarrow \{0, \dots, m\}$ such that $i \leq j$ implies $f(i) \leq f(j)$, for $0 \leq i, j \leq n$).

Recall that a contravariant functor from a category \mathcal{C} to a category \mathcal{D} is a covariant functor from the opposite category \mathcal{C}^{op} (whose morphisms are reversed) to \mathcal{D} . That is, a contravariant functor is a functor which reverses the directions of the morphisms. See [6, Ch. 2.3] for an overview of categories and functors.

The central definition of this paper is the following.

Definition 2. A *simplicial set* is a contravariant functor $\mathbf{\Delta} \rightarrow \mathbf{Set}$.

It is sometimes profitable to generalize to arbitrary categories \mathcal{C} and define a *simplicial object* in \mathcal{C} as a contravariant functor $\mathbf{\Delta} \rightarrow \mathcal{C}$. The phrase “object in \mathcal{C} ” is replaced by a reasonable name when appropriate, leading to simplicial objects beyond simplicial sets, such as simplicial groups, which will be discussed when we get to homology of simplicial sets.

A fact which greatly aids in describing a simplicial object is Proposition 5, which says that any morphism in the category $\mathbf{\Delta}$ is the composite of *coface maps* $d^i : \{0, \dots, n\} \rightarrow \{0, \dots, n+1\}$ and *codegeneracy maps* $s^i : \{0, \dots, n+1\} \rightarrow \{0, \dots, n\}$, for $0 \leq i \leq n$. The names of these two kinds of maps come from the names of the corresponding face maps and degeneracy maps on simplices which will be recalled later in this section.

The coface map d^i is defined to be the unique order-preserving bijection $\{0, \dots, n\} \rightarrow \{0, \dots, \hat{i}, \dots, n+1\}$, where the notation \hat{i} means i is omitted from $\{0, \dots, n+1\}$. That is, $d^i(j) = j$ if $0 \leq j < i$, and otherwise $d^i(j) = j+1$. The codegeneracy map s^i is defined to be the map which duplicates i , which is to say $s^i(j) = j$ if $0 \leq j \leq i$, and $s^i(j) = j-1$ if $i < j \leq n$.

Example 3. Objects in $\mathbf{\Delta}$ have a geometric realization (which should not be confused with the realization of a simplicial set) given by the covariant functor $\{0, \dots, n\} \mapsto |\Delta^n|$, where $|\Delta^n|$ is the standard n -dimensional simplex in \mathbf{Top} given by

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1 \text{ and } t_i \geq 0\},$$

and morphisms $f_* : |\Delta^n| \rightarrow |\Delta^m|$ induced by $f : \{0, \dots, n\} \rightarrow \{0, \dots, m\}$ are defined by

$$f_*(t_0, \dots, t_n) = (s_0, \dots, s_m), \quad s_j = \sum_{f(i)=j} t_i.$$

That is, the i th vertex of $|\Delta^n|$ is sent to the $f(i)$ th vertex of $|\Delta^m|$, and the barycentric coordinates are mapped linearly. It is left to the reader to show that this is, in fact, a functor $\mathbf{\Delta} \rightarrow \mathbf{Top}$. We see that the coface map d_*^i sends $|\Delta^n|$ to the i th face of $|\Delta^{n+1}|$ and that the codegeneracy map s_*^j sends $|\Delta^n|$ to $|\Delta^{n-1}|$ by collapsing together vertices j and $j+1$.

As in [4, Ch. VII.5], we have the following decomposition lemma, which we use in Proposition 5 to show the morphisms in $\mathbf{\Delta}$ are the composites of all d^i and s^j .

Lemma 4. *In $\mathbf{\Delta}$, any morphism $f : \{0, \dots, n\} \rightarrow \{0, \dots, n'\}$ has a unique representation*

$$f = d^{i_k} \dots d^{i_1} s^{j_1} \dots s^{j_h}, \quad (1)$$

where the non-negative integers h, k satisfy $n + k - h = n'$ and the subscripts i and j satisfy

$$n' \geq i_k > \dots > i_1 \geq 0, \quad 0 \leq j_1 < \dots < j_h < n.$$

Proof. A monotonically non-decreasing function f is determined by its image in $\{0, \dots, n'\}$ and by those $j \in \{0, \dots, n-1\}$ for which f does not increase (that is, j such that $f(j) = f(j+1)$). Let i_1, \dots, i_k , in increasing order, be the elements of $\{0, \dots, n'\}$ which are not in the image of f , and let j_1, \dots, j_h be the elements of $\{0, \dots, n\}$ where f does not increase. It follows that both sides of equation (1) are equal. \square

Thus, any composite of two d^i 's or s^j 's may be put into the form (1). This fact along with the definitions of d^i and s^j give the following identities, whose verification is left to the reader.

$$d^i d^j = d^{j+1} d^i \quad i \leq j, \quad (2)$$

$$s^j s^i = s^i s^{j+1} \quad i \leq j, \quad (3)$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ d^{i-1} s^j & \text{otherwise.} \end{cases} \quad (4)$$

Proposition 5. *The morphisms of $\mathbf{\Delta}$ consist of all composites of all arrows d^i and s^j subject to the relations (2), (3), and (4).*

Proof. By Lemma 4, any morphism in $\mathbf{\Delta}$ is a composite of d^i 's and s^j 's, and the relations (2), (3), and (4) suffice to put any composite of d^i 's and s^j 's into the form (1). \square

In the opposite category $\mathbf{\Delta}^{\text{op}}$, we label the morphisms corresponding to d^i and s^j by d_i and s_j , respectively. In category theoretic spirit, d_i is called a *face map* and s_j is called a

degeneracy map. Since $(-)^{\text{op}}$ is a contravariant functor, we have the following identities from relations (2), (3), and (4):

$$d_j d_i = d_i d_{j+1} \quad i \leq j, \quad (5)$$

$$s_i s_j = s_{j+1} s_i \quad i \leq j, \quad (6)$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j, j + 1 \\ s_j d_{i-1} & \text{otherwise,} \end{cases} \quad (7)$$

and, corresponding to Proposition 5, we have that all morphisms in Δ^{op} are generated by composites of d_i and s_j , subject to relations (5), (6), and (7).

Thus, a contravariant functor from Δ is completely specified by where it sends the objects $\{0, \dots, n\}$ and by where it sends the d_i and s_j . This gives us a more concrete definition for simplicial sets (and, likewise, simplicial objects by replacing “sets” with “objects from \mathcal{C} ”). To simplify notation, we write $X_n = X\{0, \dots, n\}$ and confuse d_i and s_j with Xd_i and Xs_j , respectively, when given a simplicial object X .

Proposition 6. *A simplicial set X is a collection of sets X_n for each $n \geq 0$ together with functions $d_i : X_n \rightarrow X_{n-1}$ and $s_i : X_n \rightarrow X_{n+1}$, for all $0 \leq i \leq n$ and for each n , satisfying the following relations:*

$$d_i d_j = d_{j-1} d_i \quad i < j, \quad (8)$$

$$s_i s_j = s_{j+1} s_i \quad i \leq j, \quad (9)$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j, j + 1 \\ s_j d_{i-1} & \text{otherwise.} \end{cases} \quad (10)$$

This is the standard way to write down the data of a simplicial set according to [2]. The elements of X_0 are called the *vertices* of the simplicial set, and elements of all X_n are called *simplices*. A simplex x is *degenerate* if x is in the image of some s_j .

Example 7. The standard Δ -simplex $[0, 1, \dots,]$ of the simplex $|\Delta^n|$ has a representation as a simplicial set, which we write as Δ^n . The data for the simplicial set are as follows:

$$\Delta_m^n = \{[k_0, k_1, \dots, k_m] \mid 0 \leq k_0 \leq \dots \leq k_m \leq n\}$$

$$d_i[k_0, \dots, k_m] = [k_0, \dots, \hat{k}_i, \dots, k_m]$$

$$s_j[k_0, \dots, k_m] = [k_0, \dots, k_j, k_j, \dots, k_m].$$

The relations (8), (9), and (10) are straightforward to check. For instance, $d_i d_j[k_0, \dots, k_m]$ for $i < j$ is $[k_0, \dots, \hat{k}_i, \dots, \hat{k}_j, \dots, k_m]$, and this is $d_{j-1} d_i$ since removing k_i first shifts the index of k_j by one. Checking the other relations is done in a similar manner and is left to the reader.

This example illustrates where the names of the maps come from. The map d_i gives the i th face of a simplex, while s_j gives the j th degeneracy (as in, $s_j[0, 1, \dots, n] = [0, \dots, j, j, \dots, n]$ is the $(n + 1)$ -simplex which is degenerate by doubling the j -th vertex). For instance, Δ_n^n contains $[0, \dots, n]$, and the $d_i[0, \dots, n]$ are all the standard faces of the simplex. We denote this $[0, \dots, n]$ by E_n . Note that this simplicial set is essentially the result of taking the smallest simplicial set which contains E_n as an n -simplex and which is closed under the expected face and degeneracy maps. When we look at the geometric realization functor, we will see that this definition of Δ^n is, indeed, a reasonable choice.

An alternative definition from [2, Ch. 1] is that Δ^n is the contravariant functor given by $\text{hom}_\Delta(-, \{0, \dots, n\}) : \Delta \rightarrow \mathbf{Set}$, which takes maps $f : \{0, \dots, m\} \rightarrow \{0, \dots, m'\}$ from Δ to g defined by $g(\sigma'(i)) = (\sigma \circ f)(i)$, where $\sigma \in \text{hom}_\Delta(\{0, \dots, m\}, \{0, \dots, n\})$ and $\sigma' \in \text{hom}_\Delta(\{0, \dots, m'\}, \{0, \dots, n\})$.

Example 8. As an illustration of the degenerate simplices which are present, the simplicial set Δ^0 contains one element in each Δ_m^0 , namely $[0, \dots, 0]$ with m zeros.

As these examples indicate, there are many degenerate simplices in the data for a simplicial complex, and writing all these data can be unwieldy. However, the following proposition, modified from [1], shows that we only need to specify the nondegenerate simplicies since the degenerate simplicies are the images of composites of s_j 's under the relation (9).

Proposition 9. *For a simplex z in a simplicial set X there is a unique representation*

$$z = s_{j_h} \cdots s_{j_1} x$$

where x is a nondegenerate simplex in X and the subscripts j satisfy $j_1 < \cdots < j_h$.

Proof. Suppose z is a degenerate simplex. Then $z = s_{i_1} x_1$ for some simplex x in X . By induction, if x_j is degenerate, we replace it with $s_{i_{j_1}} x_{j+1}$, and therefore there is a representation $z = s_{i_1} \cdots s_{i_h} x$ for some nondegenerate x .

Suppose z has two representations $z = Sx$ and $z = S'x'$ for composites of degeneracy maps S and S' and nondegenerate simplices x and x' . If $S = s_{i_1} \cdots s_{i_k}$, let $D = d_{i_k} \cdots d_{i_1}$. By (10), $DS = 1$, thus $x = DS'x'$. Let \tilde{D} and \tilde{S}' be the result of applying (10) repeatedly so that $DS' = \tilde{S}'\tilde{D}$ where \tilde{D} is a composite of face maps and \tilde{S}' is a composite of degeneracy maps. Since x is nondegenerate and $x = \tilde{S}'\tilde{D}x'$, it must be the case that $\tilde{S}' = 1$. Thus, x is a face of x' . By symmetry, x' is also a face of x , so it follows that $x = x'$.

The uniqueness of S and the condition on the subscripts follow from Lemma 4 when dualized. \square

Thus, in our examples above, Δ^n can be described by $[0, \dots, n]$ along with its faces (and faces' faces, etc.), as in the traditional description of a Δ -complex.

A map of simplicial sets $f : X \rightarrow Y$ is a natural transformation of the functors X and Y . These are the morphisms of the category \mathbf{S} of simplicial sets. Unraveling the definition of natural transformation, and tying into the second definition of a simplicial set, another way to write f is as a sequence of functions $f_n : X_n \rightarrow Y_n$ for each $n \geq 0$ such that $f_{n-1}d_i = d_i f_n$ and $f_{n+1}s_j = s_j f_n$.

3 Basic constructions

This section gives several basic constructions on simplicial sets which are analogous to constructions on topological spaces, as in [3]. The degree to which some of these constructions hold as analogues is examined in Section 4.

The *cartesian product* of simplicial sets is the categorical product in \mathbf{S} . Explicitly, given simplicial sets X and Y , the product $X \times Y$ is given by

$$\begin{aligned}(X \times Y)_n &= X_n \times Y_n \\ d_i(x, y) &= (d_i x, d_i y) \\ s_i(x, y) &= (s_i x, s_i y).\end{aligned}$$

The relations (8), (9), and (10) for $X \times Y$ follow from them holding for X and Y .

A *subsimplicial set* of a simplicial set X is a simplicial set Y which satisfies $Y_n \subset X_n$ for each $n \geq 0$, and which inherits the same d_i 's and s_j 's. In [3], this is also known as a *subcomplex* because simplicial sets there are called *complexes* (which is a use we avoid because “complex” is used already in use for Δ -complexes and chain complexes). A *basepoint* $*$ of X is a subsimplicial set consisting of a vertex and all its degeneracies; that is, $*$ is the inclusion of Δ^0 into X via a map of simplicial sets.

The *union* of subsimplicial sets X and Y of Z is a subsimplicial set, where $(X \cup Y)_n = X_n \cup Y_n$. The reader may verify that the resulting simplicial set is closed under the d_i and s_j since the d_i and s_j agree on each $(X \cap Y)_n$. This suggests a construction for the simplicial set of an arbitrary Δ -complex by taking a union of constituent simplicial sets Δ^n with relabeled vertices.

The *wedge product* of simplicial sets X and Y is the union of $* \times Y$ and $X \times *$ as subsimplicial sets of $X \times Y$, where $*$ is taken to be the basepoints of X and Y .

The *quotient* X/Y for Y a subsimplicial set of X is the result of identifying the simplices in Y_n together. The resulting maps d_i and s_j after the quotient continue to satisfy the relations (8), (9), and (10).

The *boundary* $\partial\Delta^n$ of Δ^n is the smallest subsimplicial set of Δ^n which contains all of the faces $d_i E_n$. While $\partial\Delta^n$ could provide a model for the $(n-1)$ -sphere, we may be more economical with simplices and define the n -sphere S^n to be the quotient $\Delta^n/\partial\Delta^n$. This S^n has two nondegenerate simplices: $*$ $\in S_0^n$ and $\sigma_n \in S_n^n$, which is the image of E_n via the quotient. We note that $d_i \sigma_n$ is a degeneracy of $*$, and we will see that the geometric realization of S^n identifies these faces with the realization of $*$.

4 Realization

In this section, we describe the realization functor $|-| : \mathbf{S} \rightarrow \mathbf{Top}$, and we also describe the functor $S : \mathbf{Top} \rightarrow \mathbf{S}$ which is right adjoint to $|-|$ and which connects singular homology to the theory of simplicial sets. We follow [3], but with more modern notation.

It is somewhat unfortunate that we are overloading our notation $|-|$, since we already used it for the covariant functor from $\mathbf{\Delta}$ to \mathbf{Top} , but this is the practice in [1] and [2]. Positively, it ends up being the case that $|\Delta^n|$ as the image of the $\mathbf{\Delta} \rightarrow \mathbf{Top}$ functor is homeomorphic to $|\Delta^n|$ as the image of Δ^n via the realization functor, so the notation is deliberately confused.

Definition 10. Let X be a simplicial set, and give each X_n the discrete topology. Let $|\Delta^n|$ be the standard n -dimensional simplex in \mathbb{R}^{n+1} . The *realization* $|X|$ of X is given by

$$|X| = \left(\prod_{n=0}^{\infty} X_n \times |\Delta^n| \right) / (\sim),$$

where $|X|$ is given the quotient topology, and where (\sim) is an equivalence relation defined by $(x, p) \sim (y, q)$ if either

- $d_i x = y$ and $d^i q = p$; or
- $s_j x = y$ and $s^j q = p$.

Here, d^i and s^j are the coface and codegeneracy maps induced by the functor $\mathbf{\Delta} \rightarrow \mathbf{Top}$.

We now describe the action of the realization functor so we can see it makes sense for Δ -complexes and so we can see it is reasonable to confuse the notation for $|-|$. In an intuitive sense, the equivalence relation collapses degeneracies and glues together faces. Say x is an n -simplex in a simplicial set X , and let us treat X as the simplicial set of a Δ -complex. That is, d_i and s_j have their geometric notions as face and degeneracy maps. First we will inspect the second relation: that $(x, s^j p) \sim (s_j x, p)$, where $p \in |\Delta^{n+1}|$. Recall that s^j maps $|\Delta^{n+1}|$ to $|\Delta^n|$ by collapsing the j and $(j+1)$ th vertices together, and s_j maps x to the $(n+1)$ -simplex by doubling the $(j+1)$ th vertex. This relation captures the notion that a point p in a realization $|\Delta^{n+1}|$ of $s_j x$ as ought to be collapsed via s^j since $s_j x$ is degenerate. Now for the first relation: that $(x, d^j p) \sim (d_j x, p)$, for $p \in |\Delta^{n-1}|$. Recall that d^j includes $|\Delta^{n-1}|$ onto the j th face of $|\Delta^n|$, and $d_j x$ gives the j th face of x . This relation glues points p on the boundary of x to their corresponding location in the realization $|\Delta^n|$ of x , and thus does the work of gluing the faces of the realization of x to the realizations of the faces of x .

Thus, the simplicial complex of a Δ -complex and the realization of a simplicial set of a Δ -complex are homeomorphic since they perform the same gluings, and we can essentially ignore the degenerate simplices in the simplicial set. In particular, $|\Delta^n|$ as the image of the realization functor $\mathbf{S} \rightarrow \mathbf{Top}$ is homeomorphic to the image of $\{0, \dots, n\}$ under the functor $\mathbf{\Delta} \rightarrow \mathbf{Top}$.

Example 11. The realization $|S^n|$ is homeomorphic to the topological n -sphere. Since the faces of σ_n are degeneracies of $*$, and $*$ is realized as the point in $(*, |\Delta^0|)$, the equivalence relations say that the faces $(d_i \sigma_n, |\Delta^{n-1}|)$ are all mapped to that point. This is a standard construction of the n -sphere using CW-complexes from topology.

One would hope that, for simplicial sets X and Y , that $|X \times Y|$ and $|X| \times |Y|$ under the product topology are homeomorphic. While there is a natural map $|X \times Y| \rightarrow |X| \times |Y|$ which is continuous, and bijective, the inverse is not necessarily continuous. The map is a homeomorphism if either

- Both X and Y are countable; or
- One of $|X|$ and $|Y|$ are locally finite (it is a fact proved in [2, Ch. 1] that the realization of a simplicial set is a CW-complex).

The reader is invited to look at [2, Ch. 1] or the references in [1] for a proof.

The following functor is important because it puts singular homology into the language of simplicial sets.

Definition 12. Let $S : \mathbf{Top} \rightarrow \mathbf{S}$ be the functor which sends a topological space Y to a simplicial set defined by $(SY)_n = \text{hom}_{\mathbf{Top}}(|\Delta^n|, Y)$, where d_i and s_j are defined by $d_i\sigma = \sigma d^i : |\Delta^{n-1}| \rightarrow Y$ and $s_j\sigma = \sigma s^j : |\Delta^{n+1}| \rightarrow Y$, for $\sigma \in S(Y)_n$. Note that the relations (8), (9), and (10) for SY follow from d^i and s^j satisfying the opposite relations in $\mathbf{\Delta}$.

It turns out that $|-|$ and S are adjoint functors, as described in the following theorem.

Theorem 13. *If X is a simplicial set and Y is a topological space, then*

$$\text{hom}_{\mathbf{Top}}(|X|, Y) \cong \text{hom}_{\mathbf{S}}(X, SY).$$

The proof of this is in [2, Ch. 1], and it is basically category theory.

5 Homotopy

In this section, we define homotopy in the abstract setting of simplicial sets. It is not the general case that homotopy is an equivalence relation. However, simplicial sets satisfying the Kan extension condition, which we describe, admit such an equivalence relation.

We first let I be the simplicial set Δ^1 (whose realization is a line segment), and let 0 and 1 denote the two vertices of I . Then, analogously to homotopy from algebraic topology, we give the following definition from [3].

Definition 14. Let $f, g : X \rightarrow Y$ be maps of simplicial sets for simplicial sets X and Y . We say f is *homotopic* to g if there is a simplicial map $F : X \times I \rightarrow Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, in which case we write $f \simeq g$. If $A \subset X$ is a subsimplicial set, then we write $f \simeq g \text{ (rel } A)$ if F is constant on A .

As we mentioned, we need an extra condition on simplicial sets for homotopy to be an equivalence relation. Let Λ_k^n be the smallest subsimplicial set of Δ^n which contains the faces $d_i E_n$ for all $i \neq k$ (see Example 7 for the definition of E_n).

Definition 15. A simplicial set X satisfies the *Kan extension condition* if any map of simplicial sets $f : \Lambda_k^n \rightarrow X$ extends to a map of simplicial sets $g : \Delta^n \rightarrow X$.

Example 16. Note that Δ^1 does not satisfy this condition. Let $f : \Lambda_0^2 \rightarrow \Delta^1$ be defined by $f([0, 1]) = [0, 1]$ and $f([0, 2]) = [0, 0]$. This function f is well-defined because we have specified what happens to all nondegenerate simplices of Λ_0^2 (that is, we can compute $f([0]) = f(d_1[0, 1]) = d_1f([0, 1]) = [0]$, and also $f(s_jx) = s_jf(x)$, and so on). Since $f([1]) = [1]$ and $f([2]) = [0]$, we cannot extend f , since $g([1, 2])$ is forced to be $[1, 0]$, which is not an element of Δ^1 .

Example 17. Importantly, SY , where S is the functor $\mathbf{Top} \rightarrow \mathbf{S}$, satisfies the Kan extension condition. Let $f : \Lambda_k^n \rightarrow SY$ be a map of simplicial sets. By Theorem 13, there is a corresponding continuous $\tilde{f} : |\Lambda_k^n| \rightarrow Y$. There is a continuous map $r : |\Delta^n| \rightarrow |\Lambda_k^n|$ which retracts the k th face and interior onto the remaining faces. Thus, $\tilde{f}r : |\Delta^n| \rightarrow Y$ is a continuous map which, again by Theorem 13, gives the required extension $g : \Delta^n \rightarrow SY$.

The following property that the Kan extension condition also works when Λ_k^n appears in a product will be used to show that (\simeq) is an equivalence relation.

Lemma 18. *If X and Y are simplicial sets, and Y satisfies the Kan extension condition, then a map of simplicial sets $f : X \times \Lambda_k^n \rightarrow Y$ can be extended to $G : X \times \Delta^n \rightarrow Y$.*

Sketch of proof. This amounts to repeated application of the Kan extension condition to fill in occurrences of Λ_l^m within $X \times \Lambda_k^N \subset X \times \Delta^n$ by induction on m . \square

The following lemma can also be generalized to show homotopy relative to a subsimplicial set $A \subset X$ is an equivalence relation.

Lemma 19. *If X and Y are simplicial sets, and Y satisfies the Kan extension condition, then (\simeq) is an equivalence relation on maps of simplicial sets $X \rightarrow Y$.*

Proof. For the following, let $f, g, h : X \rightarrow Y$ be maps of simplicial sets. We will proceed by checking each axiom.

Reflexivity. Let $F : X \times I \rightarrow Y$ be defined by $F = f \circ \pi_1$, where $\pi_1 : X \times I \rightarrow X$ is a projection. Thus, F is a homotopy, and $f \simeq f$.

Symmetry. Assume there is a homotopy F from f to g . Then, we may define $F' : X \times \Lambda_1^2 \rightarrow Y$ to be $F'(x, [0, 1]) = F(x, I)$ and $F'(x, [0, 2]) = f(x)$ for all $x \in X$, which is well-defined since $F'(x, [0]) = f(x)$ where the two legs meet. By the Kan extension condition, there is an extension $G' : X \times \Delta^2 \rightarrow Y$, and restricting G' to $X \times [0, 2]$ gives the required homotopy.

Transitivity. Assume that F is a homotopy from f to g and that G is a homotopy from g to h . Define $F' : X \times \Lambda_1^2 \rightarrow Y$ to be the homotopy F on the edge $X \times \{[0, 1]\}$ and the homotopy G on the edge $X \times \{[1, 2]\}$. By the Kan extension condition, F' extends to $G' : X \times \Delta^2 \rightarrow Y$, and restricting this to $X \times \{[0, 2]\}$ gives a homotopy from f to h .

We will not prove the following two theorems from [3], but they provides the link between the homotopy theory of simplicial sets and that of topological spaces:

Theorem 20. *For simplicial sets X and Y where Y satisfies the Kan extension condition, there is a bijective correspondence between homotopy classes of maps $X \rightarrow Y$ of simplicial sets and homotopy classes of continuous maps $|X| \rightarrow |Y|$.*

Theorem 21. *For topological spaces X and Y , where X is a CW complex, there is a bijection correspondence between homotopy classes of continuous maps $X \rightarrow Y$ and homotopy classes of maps $SX \rightarrow SY$ of simplicial sets.*

6 Homotopy groups

In this section, we will not prove anything, but instead briefly mention homotopy groups. We assume X is a simplicial set which satisfies the Kan extension condition so that homotopy is an equivalence relation.

There are a couple of ways to define the homotopy groups $\pi_n(X, *)$. One is as the set of homotopy classes of basepoint-preserving maps $\partial\Delta^{n+1} \rightarrow X$ of simplicial sets. This notion should be intuitive from algebraic topology since $\partial\Delta^{n+1}$ is homotopy equivalent to an n -sphere.

Some writers define homotopy groups without appealing to homotopy classes of maps and instead define homotopy classes of simplices. The following definition is from [1].

Definition 22. We say that two n -simplices $x, x' \in X_n$ are homotopic if

1. $d_i x = d_i x'$ for $0 \leq i \leq n$; and
2. there exists a simplex $y \in X_{n+1}$ such that
3. (a) $d_n y = x$,
 (b) $d_{n+1} y = x'$, and
 (c) $d_i y = s_{n-1} d_i x = s_{n-1} d_i x'$ for $0 \leq i \leq n-1$.

The homotopy group $\pi_n(X, *)$ is defined to be all n -simplices which contain the basepoint $*$ as a vertex, where the group operation involves using the Kan extension condition to get a third simplex from one.

Alternatively, there are more combinatoric definition of these sets. Interested readers are invited to read [1].

7 Homology

We first talk about homology of simplicial abelian groups, and then apply it to arbitrary simplicial sets. The material in this section comes from [3].

Let G be a abelian simplicial group. That is, each G_n is an abelian group, and there are homomorphisms $d_i : G_n \rightarrow G_{n-1}$ and $s_j : G_n \rightarrow G_{n+1}$ which satisfy the identities for simplicial objects. Interestingly, G actually is a chain complex with the boundary operator $\partial_n : G_n \rightarrow G_{n-1}$ defined by

$$\partial_n = \sum_{i=0}^n (-1)^i d_i.$$

It is straightforward to verify that $\partial^2 = 0$:

$$\begin{aligned} \partial^2 x &= \sum_{i=0}^{n-1} \left((-1)^i d_i \sum_{j=0}^n (-1)^j d_j x \right) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^n (-1)^{i+j} d_i d_j x = 0, \end{aligned}$$

by using the identity $d_i d_j = d_{j-1} d_i$ for $i < j$. We define the homology groups of this chain complex to be $H_n(G) = \ker \partial_n / \text{im } \partial_{n+1}$.

Let $Z : \mathbf{Set} \rightarrow \mathbf{Ab}$ be the functor which takes sets and gives the free abelian groups on those sets. Then, if X is a simplicial set, $Z(X)$ is a simplicial free abelian group where $(Z(X))_n$ is the free abelian group with the elements of X_n as generators.

Definition 23. The *homology groups* $H_n(X)$ of a simplicial set X are the homology groups $H_n(Z(X))$ using the boundary operator $\partial_n = \sum_{i=0}^n (-1)^i d_i$.

Example 24. For a topological space Y , it is clear that the homology groups $H_n(SY)$ are isomorphic to the singular homology groups of Y from algebraic topology since $(Z(SY))_n$ is just the singular n -chain group.

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