Solutions to Problem Set 1

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Problem 1

Statement. Let $n \geq 1$ be an integer. Let $\mathbf{CP}^n$ denote the set of all lines in $\mathbf{C}^{n+1}$ passing through the origin. There is a natural map $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$ taking a point to the line it spans. We give $\mathbf{CP}^n$ the quotient topology, so that a set U in $\mathbf{CP}^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbf{C}^{n+1}$. Let $U_i \subset \mathbf{CP}^n$ denote the set of points of the form $\pi(x_0, \ldots, x_n)$ where $x_i \neq 0$.

(a) Show that the U_i form an open cover of $\mathbf{CP}^n$.
(b) Show that an intersection of $k+1$ distinct elements of $\{U_0, \ldots, U_n\}$ is homeomorphic to $(\mathbf{C}^\times)^k \times \mathbf{C}^{n-k}$, for $0 \leq k \leq n$. (In particular, each U_i is homeomorphic to $\mathbf{C}^n$.)
(c) Prove the following lemma. Let X be a topological space and let $\mathscr{U}$ be a finite open cover of X. Suppose that each element of $\mathscr{U}$ is simply connected and any intersection of elements of $\mathscr{U}$ is non-empty and path-connected. Then X is simply connected. [Hint: use van Kampen's theorem.]
(d) Conclude that $\mathbf{CP}^n$ is simply connected.

Solution. (a) The set $\pi^{-1}(U_i)$ consists of those points $(x_0, \ldots, x_n)$ of $\mathbf{C}^{n+1}$ with $x_i \neq 0$. This is open, and so U_i is open in $\mathbf{CP}^{n+1}$. Every point of $\mathbf{CP}^n$ is of the form $\pi(x_0, \ldots, x_n)$ where at least one of $x_0, \ldots, x_n$ is not zero. If x_i is non-zero, then the point belongs to U_i. This shows that the U_i cover.

(b) The situation is symmetrical, so to ease notation we only consider the intersection $U = U_0 \cap \cdots \cap U_k$. Let V be the subspace of $\mathbf{C}^{n+1}$ consisting of elements of the form $(1, x_1, \ldots, x_n)$ where $x_i \neq 0$ for $1 \leq i \leq k$. We give V the subspace topology, with which it is clearly homeomorphic to $(\mathbf{C}^\times)^k \times \mathbf{C}^{n-k}$. It is clear that π maps V into U. We now define a map in the opposite direction:

$$i : U \to V, \qquad i(\pi(x_0, \ldots, x_n)) = \left(1, \tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}\right).$$

It is easy to see that i is well-defined: first, if $\pi(x)$ belongs to U then $x_0 \neq 0$ and so we can divide by x_0, and second, if $\pi(x) = \pi(y)$ then $x = \lambda y$ for some $\lambda \in \mathbf{C}^\times$, and so $\frac{x_i}{x_0} = \frac{y_i}{y_0}$ for all i. It is also easy to see that π and i are mutual inverses. Indeed, if $x \in V$ then $x_0 = 1$ and so $i(\pi(x)) = x$. Similarly, if $x \in U$ then we can write $x = \pi(y)$ for some $y \in \mathbf{C}^{n+1} \setminus \{0\}$. We then have $i(x) = y_0^{-1} y$, and so $\pi(i(x)) = \pi(y_0^{-1} y) = \pi(y) = \pi(x)$, since $y_0^{-1} y$ and y span the same line. It remains to show continuity of each map. The map $\pi : V \to U$ is continuous since it is just the restriction of the map $\mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$, which is continuous by definition.

We now show that i is continuous. Let W be an open set of V. We must show that $i^{-1}(W)$ is an open subset of U. By the definition of the topology on U, this set is open if and only if $\pi^{-1}(i^{-1}(W))$ is an open subset of $\mathbf{C}^{n+1}$. Now, $\pi^{-1}(i^{-1}(W))$ is easily seen to be $\mathbf{C}^\times W$, i.e., it consists of all non-zero multiples of elements of W. Let x be an element of W. We can then find an open neighborhood W_1 of 1 in $\mathbf{C}^\times$, such that $W_1 = W_1^{-1}$, and an open neighborhood W_2 of x in W such that $W_1 W_2 \subset W$. It follows then that $\mathbf{C}^\times W$ contains $W_1 \times W_2$ (where here we regard W_2 as a subset of $(\mathbf{C}^\times)^k \times \mathbf{C}^{n-k}$). This is an open set of $\mathbf{C}^{n+1} \setminus \{0\}$, which shows that x belongs to the interior of $\mathbf{C}^\times W$. Now, if x is an arbitrary element of $\mathbf{C}^\times W$, then we can find $\lambda \in \mathbf{C}^\times$ such that $\lambda x \in W$. If W' is an open neighborhood of λx in $\mathbf{C}^{n+1} \setminus \{0\}$ contained in $\mathbf{C}^\times V$, then $\lambda^{-1} W'$ is such a neighborhood of x. It follows that all points of $\mathbf{C}^\times W$ belong to its interior, i.e., it is open.

(c) [When writing this problem, I forgot that we were doing a general version of van Kampen's theorem allowing for covers with more than two open sets. The following inductive proof establishes (c) using van Kampen's theorem for only two element covers. Part (c) is fairly trivial to derive from the general van Kampen theorem.]

We proceed by induction on the cardinality of $\mathscr{U}$. It is clear if $\#\mathscr{U} = 1$. Now let $\mathscr{U} = \{U_1, \ldots, U_n\}$ be given. Put $Y = U_2 \cup U_3 \cup \cdots \cup U_n$. By the inductive hypothesis applied to Y and the cover $\{U_2, \ldots, U_n\}$, we conclude that Y is simply connected. Now, $U_1 \cap Y = (U_1 \cap U_2) \cup \cdots \cup (U_1 \cap U_n)$. We claim that this set is path connected. Thus let x and y be points in it. Then x belongs to $U_1 \cap U_i$ and y belongs to $U_1 \cap U_j$, for some i and j. The sets $U_1 \cap U_i$ and $U_1 \cap U_j$ are non-empty, path-connected, contained in Y and each contain the non-empty $U_1 \cap U_i \cap U_j$. It follows that we can find a path in $U_1 \cap U_i$ from x to some point in $U_1 \cap U_i \cap U_j$, and then a path in $U_1 \cap U_j$ from this point to y. The composite path provides a path from x to y contained entirely in $U_1 \cap Y$. This proves that $U_1 \cap Y$ is path connected. Now, U_1 and Y are simply connected and their intersection is path-connected; van Kampen's theorem now shows that $X = U_1 \cup Y$ is simply connected. This completes the proof.

(d) The open cover $\{U_0, \ldots, U_n\}$ of $\mathbf{CP}^n$ satisfies the hypotheses of the lemma from (c), and so $\mathbf{CP}^n$ is simply connected.

Problem 2

Statement. Let X be a topological space and let x_1 and x_2 be two points in X. Given a path h between x_1 and x_2, we have seen that there is a canonical isomorphism

$$i_h : \pi_1(X, x_1) \to \pi_1(X, x_2).$$

Write $C(G)$ for the set of conjugacy classes in a group G, and let

$$\bar{i}_h : C(\pi_1(X, x_1)) \to C(\pi_1(X, x_2))$$

denote the map induced by i_h.

(a) Give an example (i.e., specify X, x_1, x_2, h and h') where $i_h \neq i_{h'}$, with proof.
(b) Show that $\bar{i}_h = \bar{i}_{h'}$ for any two paths h and h'.
(c) Assume $\pi_1(X, x_1)$ is abelian. Show that $i_h = i_{h'}$ for any h and h'.

Solution. (a) Take X to be $S^1 \vee S^1$, take $x_1 = x_2$ to be the point where the two circles meet, take h to be the trival path from x_1 to itself and take h' to be the path going around one of the circles. Then i_h is the identity map from $\pi_1(X, x_1)$ to itself, while $i_{h'}$ is given by conjugation by $[h']$, regarded as an element of $\pi_1(X, x_1)$. Since the center of $\pi_1(X, x_1)$ is trivial (we know this fundamental group is the free group on two letters) and $[h']$ is non-trivial (it is one of the generators), conjugation by $[h']$ is not the identity map on $\pi_1(X, x_1)$. Thus $i_h \neq i_{h'}$.

(b) If g is an element of $\pi_1(X, x_1)$ then $i_h(g)$ is by definition $h^{-1}gh$, where h^{-1} is the reverse path from x_2 to x_1, and juxtaposition denotes concatenation of paths. We thus have

$$i_{h'}(g) = (h')^{-1}gh' = (h')^{-1}hi_h(g)h^{-1}h' = ai_h(g)a^{-1},$$

where $a = (h')^{-1}h$ is an element of $\pi_1(X, x_2)$. This shows that $i_{h'}(g)$ and $i_h(g)$ are conjugate in $\pi_1(X, x_2)$. Thus $\bar{i}_{h'} = \bar{i}_h$.

(c) In an abelian group, two elements are conjugate if and only if they are equal. Thus $\bar{i}_h = \bar{i}_{h'}$ implies $i_h = i_{h'}$.

Problem 3

Statement. Let X be a metric (and thus topological) space. Fix a basepoint x_0 in X; the word "loop" will mean "loop based at x_0" in this problem. Let ΩX denote the set of all loops in X, i.e., the set of all continuous functions $p : [0, 1] \to X$ with $p(0) = p(1) = x_0$. Define a distance function on ΩX by $d(p_1, p_2) = \max_{x \in [0,1]} d(p_1(x), p_2(x))$.

(a) Show that concatentation of loops defines a continuous map $\Omega X \times \Omega X \to \Omega X$. Conclude that there is a natural map of sets $\pi_0(\Omega X) \times \pi_0(\Omega X) \to \pi_0(\Omega X)$. [Here π_0 denotes the set of path components.]

(b) Show that two loops in X are homotopic if and only if the corresponding points of ΩX are in the same path component.
(c) Construct a canonical bijection of sets $\pi_0(\Omega X) \to \pi_1(X, x_0)$. Show that this map is a homorphism, in the sense that it respects the multiplications on the two sets (the one on $\pi_0(\Omega X)$ constructed in (a) and the usual group operation on $\pi_1(X, x_0)$).

Solution. (a) Let (p_1, p_2) be an element of $\Omega X \times \Omega X$ and let $\epsilon > 0$ be given. If (p_1', p_2') is another point of $\Omega X \times \Omega X$ such that $d(p_1, p_1') < \epsilon$ and $d(p_2, p_2') < \epsilon$, then $d(p_1 p_2, p_1' p_2') < \epsilon$ as well; this is immediate from the definitions. By elementary properties of metric spaces, this implies that concatenation of loops is continuous. We now have maps

$$\pi_0(\Omega X) \times \pi_0(\Omega X) \to \pi_0(\Omega X \times \Omega X) \to \pi_0(\Omega X),$$

where the first comes from basic point-set topology, and the second is the one induced from the concatenation map.

(b) Let p_0 and p_1 be two loops in X. Suppose first that they are homotopic. Let p_t be a homotopy between them. Then $t \mapsto p_t$ provides a path between p_0 and p_1 in ΩX, provided it is continuous. We now show that it is continuous. Let $t \in [0,1]$ and $\epsilon > 0$ be given. Since $(t, x) \mapsto p_t(x)$ is continuous, for each $x \in [0,1]$ we can find an open rectangle U_x in $[0,1]^2$ containing (t, x) with the property that $d(p_{t_1}(x_1), p_{t_2}(x_2)) < \epsilon$ for all (t_1, x_1) and (t_2, x_2) in U_x. By compactness of the interval, we can find $x_1, \ldots, x_n$ such that $U_{x_1}, \ldots, U_{x_n}$ covers $t \times [0,1]$. The union of these open sets contains a rectangle of the form $V \times [0,1]$, where V is an open interval containing t. Thus for any $t' \in V$ we have $d(p_t, p_{t'}) < \epsilon$. This shows that $t \mapsto p_t$ is continuous.

Now suppose that p_0 and p_1 belong to the same path component of ΩX. Let $P : [0,1] \to \Omega X$ be a path connecting them, i.e., a continuous map with $P(0) = p_0$ and $P(1) = p_1$. Let $e : [0,1] \times \Omega_X \to X$ be the evaluation map $(x, p) \mapsto p(x)$. Define a map $[0,1]^2 \to X$ by $(t, x) \mapsto e(x, P(t))$. This is a homotopy between p_0 and p_1, provided it is continuous. To show that it is continuous, it suffices to show that e is continuous.

We now do this. Let $(x, p) \in [0,1] \times \Omega_X$ and $\epsilon > 0$ be given. Let J be an open neighborhood of x such that $d(p(x_1), p(x_2)) < \epsilon$ for all $x_1, x_2 \in J$. Let U be the open ball in ΩX centered at p and of radius ϵ. If (x_1, p_1) belongs to $J \times \Omega X$ then

$$d(p(x), p_1(x_1)) \leq d(p(x), p(x_1)) + d(p(x_1), p_1(x_1)) \leq 2\epsilon.$$

This shows that e is continuous.

(c) Define a map $i : \pi_0(\Omega X) \to \pi_1(X, x_0)$ as follows. Let C be a path component of ΩX and let p be a point on C. Then $i(C)$ is the class of p in $\pi_1(X, x_0)$. This is well-defined by (b): if p' is a different point on C, then there is a path between p and p' in ΩX and thus a homotopy between p and p' in X, and so p and p' represent the same class in $\pi_1(X, x_0)$. It is also injective by (b). It is obviously surjective. Furthermore, it is obviously compatible with the two product operations, since they're both defined by concatenation.

Problem 4

Statement. In this problem, we will show that every finitely presented group occurs as a fundamental group.

(a) Let G be a group, let a be an element of G and let N be the normal closure of the subgroup generated by a. [Explicitly, N is the subgroup of G generated by all conjugates of a.] Let $\mathbf{Z} \to G$ be the map defined by $1 \mapsto a$. Show that the amalgamated free product $G *_{\mathbf{Z}} 1$ is isomorphic to G/N. [Here 1 denotes the trivial group.]
(b) Let X be a topological space with base point x_0 and let $i : S^1 \to X$ be a loop based at x_0. Let X' be the topological space obtained by attaching a 2-disc to X via i; that is, X' is the quotient of $X \amalg D^2$ where an element $x \in S^1 = \partial D^2$ is identified with $i(x) \in X$. Show that

$\pi_1(X', x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by the class of i. [Hint: use van Kampen's theorem.]

(c) Show that every finitely presented group occurs as a fundamental groups. [Hint: let G be a finitely presented group. Pick a presentation. Start with a bouquet of circles, one for each generator. Attach a 2-disc for each relation and apply (b).]

Solution. (a) It's easiest to prove this using the universal property of amalgamated free products. Let H be an arbitrary group. Giving a map $G *_{\mathbf{Z}} 1 \to H$ is the same as giving a map $G \to H$ that kills a, and this is the same as giving a map $G/N \to H$. This shows that G/N satisfies the same universal property as $G *_{\mathbf{Z}} 1$, and so the two are isomorphic.

(b) We first remark that it suffices to treat the case where X is path-connected. Indeed, let X_1 be the path component to which x_0 belongs and let X_1' be constructed in an analogous manner to X'. We have a diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r] & \pi_1(X', x_0) \\ \pi_1(X_1, x_0) \arrow[r] \arrow[u] & \pi_1(X_1', x_0) \arrow[u] \end{tikzcd}$$

The diagram obviously commutes. The vertical maps are easily seen to be isomorphisms, since π_1 only depends on the path component that the basepoint lies in. We thus see that if the bottom map is surjective with kernel the normal closure of $[i]$, then the top map has the same property. Thus we may as well replace X by X_1 and assume that X is path-connected.

Let 0 be a chosen point on D^2 not on its boundary. Let $U = X' \setminus \{0\}$ and let V be the open unit disc, regarded as a subset of X'. Let x_1 be a point in $U \cap V$ and let h be a path from x_0 to x_1 such that $h(t) \in U \cap V$ for $t \neq 0$. We have a commutative diagram

$$\begin{tikzcd} \pi_1(U, x_1) \arrow[r] & \pi_1(X', x_1) \\ \pi_1(U, x_0) \arrow[r] \arrow[u] & \pi_1(X', x_0) \arrow[u] \end{tikzcd}$$

where the horizontal maps are the natural ones and the vertical ones are i_h. Furthermore, the natural map $\pi_1(X, x_0) \to \pi_1(U, x_0)$ is an isomorphism, since U deformation retracts onto X. It follows that $\pi_1(X, x_0) \to \pi_1(X', x_0)$ is a surjection with kernel the normal closure of the subgroup generated by $[i]$ if and only if $\pi_1(U, x_1) \to \pi_1(X', x_1)$ is a surjection with kernel the normal closure of the subgroup generated by $j = i_h([i])$. (Sorry for the two i's!)

Now, U and V are path-connected open sets that cover X' and their intersection is path connected and contains x_1. In fact, their intersection is an annulus and j generates its fundamental group. By van Kampen's theorem, $\pi_1(X', x_1) = \pi_1(U, x_1) *_{\mathbf{Z}} 1$, where $\mathbf{Z}$ is really $\pi_1(U \cap V, x_1)$ and 1 is really $\pi_1(V, x_1)$. Since the map $\mathbf{Z} \to \pi_1(U, x_1)$ sends 1 to j, we see from part (a) that $\pi_1(X', x_1)$ is the quotient of $\pi_1(U, x_1)$ by the normal subgroup generated by j. This completes the proof of (b).

(c) Let G be a finitely generated group. Let $a_1, \ldots, a_n$ be generators for G and let $b_1, \ldots, b_m$ be sufficient relations to present G. Let G_0 be the free group on the a_i and let G_i be the quotient of G_0 by the normal subgroup generated by $b_1, \ldots, b_i$. Note that G_i is the quotient of G_{i-1} by the normal subgroup generated by b_i and that $G_m = G$. We now prove inductively that there are spaces

$$X_0 \to X_1 \to \cdots \to X_m$$

such that $\pi_1(X_i) = G_i$, and the map $\pi_1(X_i) \to \pi_1(X_{i+1})$ is the natural quotient map $G_i \to G_{i+1}$. To obtain X_0, simply take a bouquet of circles, one for each a_i. Assume now that we have constructed X_{i-1}. Via the map $G_0 \to G_{i-1}$, we can regard b_i as an element of $\pi_1(X_{i-1})$. By (b), we can now attach a 2-disc to X_{i-1} to obtain a space X_i with $\pi_1(X_i) = G_i$. This completes the proof.

Problem 5

Statement. Let G be a topological group; thus G is simulateneously a group and a topological space, and the multiplication map $G \times G \to G$ and inversion map $G \to G$ are continuous.

(a) Show that there is a unique group structure on $\pi_0(G)$ such that the natural map $G \to \pi_0(G)$ is a group homomorphism.

(b) Show that $\pi_1(G, 1)$ is a commutative group. [Hint: if c is a loop in G based at 1 and g is an element of G then $t \mapsto gc(t)$ is a loop in G based at g. Using this you can slide one loop along another to show that they commute in π_1.]

Solution. (a) The map $G \to \pi_0(G)$ is surjective, so there is at most one group structure on $\pi_0(G)$ which makes this map a homomorphism. Let G° be the path component of G containing the identity element. Then G° is a normal subgroup of G. Indeed, suppose that x and y belong to G°. Let p be a path from 1 to x and let q by a path from 1 to y. Then $t \mapsto xq(t)$ is a path from x to xy. Concatenating this with p, we obtain a path from 1 to xy. This shows that G° is closed under multiplication. It is clear that G° is closed under inversion; $t \mapsto p(t)^{-1}$ provides a path from 1 to x^{-1}. This shows that G° is a group. Finally, if y is any element of G then $t \mapsto yp(t)y^{-1}$ is a path from 1 to yxy^{-1}, and so G° is normal.

We now claim that two elements x and y of G belong to the same path component if and only if $xy^{-1} \in G^\circ$. First suppose that $xy^{-1} \in G^\circ$. Let p be a path from 1 to xy^{-1}. Then $t \mapsto p(t)y$ is a path from x to y, and so x and y lie in the same path component. Conversely, suppose that p is a path from x to y. Then $t \mapsto p(t)y^{-1}$ is a path from xy^{-1} to 1, and so xy^{-1} belongs to G°. This establishes the claim.

It follows that the natural map $G \to \pi_0(G)$ factors as $G \to G/G^\circ$ followed by the bijection $G/G^\circ \to \pi_0(G)$. Since G° is normal, G/G° is a group, and the bijection of this with $\pi_0(G)$ gives a group structure on $\pi_0(G)$.

(b) Let f and g be two loops based at the identity. Let F_t be the concatenation of the following paths: first, the path $f|_{[0,t]}$, from 1 to $f(t)$; then, the loop $s \mapsto f(t)g(s)$, based at $f(t)$; and finally, the map $f|_{[t,1]}$ from $f(t)$ to 1. (In the second step, the juxtaposition denotes multiplication in the group.) One easily sees that $F : [0,1]^2 \to G$ is continuous. (The function F is defined piecewise on three regions in $[0,1]^2$. It is clearly continuous on each region, and there is agreement at the boundaries. This implies it is continuous.) Now, $F(0)$ is the concatenation gf, while $F(1)$ is the concatenation fg. Thus fg and gf are homotopic, and so $\pi_1(X, 1)$ is commutative.

Problem 6

Statement. Let $G = \mathrm{SL}(2, \mathbf{R})$, the group of 2×2 real matrices with determinant 1. We can naturally regard G as a closed subset of $\mathbf{R}^4$, and thus (after a few simple verifications) as a topological group. Let $B^\circ \subset G$ be the subgroup of matrices which are upper-triangular with positive entries on the diagonal. Let $K \subset G$ be the subgroup of rotations matrices. [An element of G belongs to K if and only if its two columns form an orthonormal basis of $\mathbf{R}^2$.]

(a) Show that B° is homeomorphic to $\mathbf{R}^2$, and is thus contractible.

(b) Show that K is homeomorphic to S^1.

(c) Show that the map $B^\circ \times K \to G$ sending (b, k) to bk is a homeomorphism.

(d) Conclude that G is homotopy equivalent to S^1, and thus has fundamental group $\mathbf{Z}$.

Solution. (a) The group B° consists of matrices of the form

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}$$

with $a > 0$. We thus have evident bijections between B° and $\mathbf{R}_{\geq 0} \times \mathbf{R}$ sending a point (a, b) in $\mathbf{R}_{\geq 0} \times \mathbf{R}$ to the above matrix, and the above matrix to (a, b) in $\mathbf{R}_{\geq 0} \times \mathbf{R}$. These maps are each continuous since their components are. We thus find that B° is homeomorphic to $\mathbf{R}_{\geq 0} \times \mathbf{R}$. Since $\mathbf{R}_{\geq 0}$ is homeomorphic to $\mathbf{R}$ (by the logarithm and exponential maps), we find that B° is homeomorphic to $\mathbf{R}^2$, and thus contractible.

(b) A rotation matrix necessarily has the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

with $x^2+y^2=1$. We thus have evident bijections between K and S^1 sending a point (x, y) on S^1 to the above matrix, and the above matrix to the point (x, y) on S^1. These maps are each continuous since their components are continuous functions.

(c) Let $\langle , \rangle$ be the standard inner product on $\mathbf{R}^2$; it is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2.$$

Let $\|x\|^2 = \langle x, x \rangle$ be the associated norm. Let e_1, e_2 be the standard basis for $\mathbf{R}^2$. Let g be an element of G. Then ge_1 and ge_2 is also a basis for $\mathbf{R}^2$. The Graham–Schmit process allows us to take this basis and obtain an orthonormal basis. Precisely, put

$$f_1 = \frac{ge_1}{\|ge_1\|} = A(g)ge_1, \qquad f_2 = \frac{ge_2 - \langle ge_1, ge_2 \rangle \|ge_1\|^{-1} ge_1}{\|ge_2 - \langle ge_1, ge_2 \rangle \|ge_1\|^{-1} ge_1\|} = D(g)ge_2 + B(g)ge_1$$

(Here $A(g)$, $B(g)$ and $D(g)$ are just real numbers; for instance, $A(g) = \|ge_1\|^{-1}$.) Then f_1 and f_2 form an orthonormal basis for $\mathbf{R}^2$. Let $\beta(g)$ be defined by

$$\beta(g)^{-1} = \begin{pmatrix} A(g) & B(g) \\ & D(g) \end{pmatrix}$$

Then $f_i = g\beta(g)^{-1}e_i$. Since $\kappa(g) = g\beta(g)^{-1}$ takes the orthonormal basis (e_1, e_2) to the orthonormal basis (f_1, f_2), it follows that $\kappa(g)$ belongs to K. Thus $\beta(g)$ has determinant 1, and is clearly upper triangular, and so belongs to B°. Since the components of $\beta(g)^{-1}$ (i.e., A, B and D) are clearly continuous functions of G, we find that $\beta : G \to B^\circ$ is continuous. Since κ is defined from β and matrix multiplication, $\kappa : G \to K$ is continuous.

We have thus constructed a continuous function

$$G \to K \times B^\circ, \qquad g \mapsto (\kappa(g), \beta(g))$$

which is a one-sided inverse to the (obviously) continuous function

$$K \times B^\circ \to G, \qquad (b, k) \mapsto bk,$$

i.e., the composite $G \to K \times B^\circ \to G$ is the identity. To finish the proof, it suffices to show that the map $K \times B^\circ \to G$ is injective, for then the two maps are forced to be mutual inverses. Thus assume that $bk = b'k'$. Then $(b')^{-1}b = k'k^{-1}$, and so $k'k^{-1}$ belongs to $K \cap B^\circ$. However, $K \cap B^\circ = 1$ (easy calculation), and so $k = k'$, from which it follows that $b = b'$. This completes the proof.

[I just noticed that I did things backwards! The problem asked to show that $B^\circ \times K \to G$ is a homeomorphism and I showed that $K \times B^\circ \to G$ is a homeomorphism. There are two ways to fix this. First, one could change the above proof, using row operations instead of column operations. Or, one could observe that there is a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] & G \\ K \times B^\circ \arrow[u] \arrow[r, "\psi"] & B^\circ \times K \arrow[u] \end{tikzcd}$$

where the vertical maps are multiplication maps, $\phi(g) = g^{-1}$ and $\psi(k, b) = (b^{-1}, k^{-1})$. Since ϕ, ψ and the left map are homeomorphisms, it follows that the right map is as well.]

(d) From (a)–(c), we find that G is homeomorphic to $\mathbf{R}^2 \times S^1$, and thus homotopy equivalent to S^1. Thus $\pi_1(G) = \mathbf{Z}$.