18.786 Midterm Exam

April 1, 2010

Solve **two** out of these three problems. Justify your answers: for instance, saying that a high-level gp function (such as bnfclgp()) outputs the answer you give is not enough justification. You may use gp without justification to do elementary calculations such as computing the discriminant of a polynomial, determinant of a matrix, doing arithmetic modulo primes or a polynomial, etc.

You may use results proved in class, lecture notes, or the problem sets. You are not allowed to google/wiki search for answers.

1. Let $f(x) = x^3 + x - 4$, and $K = \mathbb{Q}[x]/(f(x))$ be the number field obtained by adjoining a root α of f.
   (a) Find out the number of real and complex embeddings of K.
   (b) Show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
   (c) Find out how the primes $2, 3, 5$ factor in K.
   (d) Calculate the class group of K.
2. Let p be a prime.
   (a) Describe and count all the (finitely many, by homework) quadratic extensions of $\mathbb{Q}_p$ (e.g. by describing polynomials $f(x)$ such that the extension is $\mathbb{Q}_p[X]/(f(x))$). Describe the ramification behaviour of p in each of these extensions.
   (b) Recall that the valuation on $\mathbb{Q}_p$ extends uniquely to any finite extension, and hence to the algebraic closure $\overline{\mathbb{Q}}_p$. Show that $\overline{\mathbb{Q}}_p$ is not a complete field for this valuation.
3. Let $g(x) = x^4 + 18x^2 + 2$, and $K = \mathbb{Q}[x]/(g(x))$ be the number field obtained by adjoining a root α of g.
   (a) Compute the number of real and complex embeddings of K, and the rank of the unit group of $\mathcal{O}_K$.
   (b) Compute the torsion part of the units. (Hint: if $\zeta_n \in K$, compare the primes which ramify in K and in $\mathbb{Q}(\zeta_n)$.)
   (c) Describe an explicit non-torsion unit. (Hint: find a real quadratic subfield of K.)