ALGEBRAIC NUMBER THEORY

LECTURE 8 NOTES

1. Section 4.1

We say a set $S \subset \mathbb{R}^n$ is discrete if the topology induced on S is the discrete topology. Check that this is equivalent to the definition in the book (every compact subset K of $\mathbb{R}^n$ intersects S in a finite set).

A lattice is a discrete subgroup Λ of $\mathbb{R}^n$ of rank n as a $\mathbb{Z}$-module.

Proof of Minkowski's theorem. Translate all the parts of S to a fixed fundamental parallelopiped. Since the sum of volumes of the translates is larger than the volume of the fundamental domain, two of the portions must overlap. The difference of corresponding points in two overlapping portions gives a vector $x - y$ in the lattice, for $x, y \in S$. □

For a number field K of degree $n = r_1 + 2r_2$ over $\mathbb{Q}$, it follows from $\dim_{\mathbb{Q}} K = n = \dim_{\mathbb{R}} \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ that $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, although non-canonically (there are choices made in the canonical embedding!)

We'll see a version for tensoring with the non-archimedean completions of $\mathbb{Q}$ later.

2. Section 4.3

Example. Consider the field $K = \mathbb{Q}(\alpha)$, where α is the unique real root of $X^3 - X - 1$. We calculate that $\mathbb{Z}[\alpha]$ has discriminant -23, which is square-free. This implies that $\mathcal{O}_K = \mathbb{Z}[\alpha]$, since otherwise the (absolute) discriminant of $\mathcal{O}_K$ would be $23/N^2$ for some integer $N > 1$, and could not be an integer, which is impossible.

Now let's determine the class group. We have $r_1 = r_2 = 1$. By the proof of Theorem 2, we need only look at ideals $\mathfrak{a}$ of norm $N\mathfrak{a} \leq \frac{4}{\pi} \frac{3!}{27} \sqrt{23} \approx 1.357$. Since the norm is an integer, it must be 1. So $\mathfrak{a} = \mathcal{O}_K$ which is principal. Therefore the class group is trivial and $\mathcal{O}_K$ is a PID.

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