LECTURE 8

Tate Cohomology and Inverse Limits

Recall that, for an extension L/K of local fields with Galois group $G := \mathbb{Z}/n\mathbb{Z}$, we were trying to show that $\#\hat{H}^0(G, L^\times) = n$. We claimed that $\chi(L^\times) = n$ if and only if $\chi(\mathcal{O}_L^\times) = 1$, where χ denotes the Herbrand quotient $\#\hat{H}^0/\#\hat{H}^1$ and we recall that a finite group has Herbrand quotient equal to 1 and that χ is multiplicative for short exact sequences.

Last time, we proved $\chi(\mathcal{O}_L) = 1$, using the normal basis theorem to show that $L \simeq K[G]$, that $\mathcal{O}_L$ contains a finite-index open subgroup Γ such that $\mathcal{O}_L^\times \supset \Gamma \simeq \mathcal{O}_L^+$ via G-equivalence (so that Γ is closed under the G-action), and that $\mathcal{O}_K[G] \simeq \Gamma \subseteq \mathcal{O}_L$. We then used Claim 7.10 to show that $\hat{H}^i(G, \Gamma) = 0$ for each i, hence $\chi(\mathcal{O}_L) = \chi(\Gamma) = 1$.

Now we'd like to give a better, i.e., more algebraic (without p-adic exponentials!), proof that $\chi(\mathcal{O}_L^\times) = 1$. So fix some open subgroup $\Gamma \subseteq \mathcal{O}_L$ isomorphic to $\mathcal{O}_K[G]$ (as $\mathcal{O}_K[G]$-modules).

Claim 8.1. *For sufficiently large N, $1 + \mathfrak{p}_K^N\Gamma$ is a subgroup of $\mathcal{O}_L^\times$, where $\mathfrak{p}_K$ is the maximal ideal of $\mathcal{O}_K$.*

Proof. For $x, y \in \Gamma \subseteq \mathcal{O}_L$, we have

$$(1 + \pi^N x)(1 + \pi^N y) = 1 + \underbrace{\pi^N(x + y)}_{\in \mathfrak{p}_K^N\Gamma} + \underbrace{\pi^{2N}(xy)}_{\in \mathfrak{p}_K^{2N}\mathcal{O}_L}, \tag{8.1}$$

where π is a uniformizer of $\mathfrak{p}_K$. Thus, if we choose N large enough that $\mathfrak{p}_K^N\mathcal{O}_L \subseteq \Gamma$, which is possible because Γ is an open subgroup of $\mathcal{O}_L$, this product will be in $1 + \pi^N\Gamma$ and therefore $1 + \mathfrak{p}_K^N\Gamma$ will be a subgroup of $\mathcal{O}_L^\times$. $\square$

Claim 8.2. *The cohomologies of Γ all vanish.*

Choose N such that $\mathfrak{p}_K^N\mathcal{O}_L \subseteq \mathfrak{p}_K\Gamma$. Then the last term in (8.1) is in $\mathfrak{p}_K^{2N}\mathcal{O}_L \subseteq \mathfrak{p}_K^{N+1}\Gamma$. This suggests that we ought to filter $1 + \mathfrak{p}_K^N\Gamma$ with additive subquotients, that is, by $1 + \mathfrak{p}_K^{N+i}\Gamma$, so that

$$(1 + \mathfrak{p}_K^{N+i}\Gamma)/(1 + \mathfrak{p}_K^{N+i+1}\Gamma) \simeq \Gamma/\mathfrak{p}_K\Gamma \simeq k_K[G]$$

for all $i \geq 0$ as additive groups by the above calculation, where k_K denotes the residue field of K. Moreover, these isomorphisms are *Galois-equivariant*, or *G-equivariant*, as the G-action preserves all terms (Γ is preserved by assumption), hence acts on both sides, and is preserved by the isomorphism. Thus, by Claim 7.10,

$$\hat{H}^j(G, (1 + \mathfrak{p}_K^{N+i}\Gamma)/(1 + \mathfrak{p}_K^{N+i+1}\Gamma)) = \hat{H}^j(G, k_K[G]) = 0,$$

for each $i \geq 0$ and j. As a corollary, for which we need the following lemma,

$$\hat{H}^j(G, (1 + \mathfrak{p}_K^N\Gamma)/(1 + \mathfrak{p}_K^{N+i}\Gamma)) = 0 \tag{8.2}$$

for all $i \geq 0$ and j.

LEMMA 8.3. *For any short exact sequence*

$$0 \to M \to E \to N \to 0$$

of G-modules, $\hat{H}^i(G, M) = \hat{H}^i(G, N) = 0$ implies $\hat{H}^i(G, E) = 0$ for each i.

PROOF. By (6.4), we have an exact sequence

$$\underbrace{\hat{H}^i(G, M)}_{0} \xrightarrow{\alpha} \hat{H}^i(G, E) \xrightarrow{\beta} \underbrace{\hat{H}^i(G, N)}_{0},$$

hence $\hat{H}^i(E) = \mathrm{Ker}(\beta) = \mathrm{Im}(\alpha) = 0$, as desired. □

Now, we have an exact sequence

$$0 \to \frac{1 + \mathfrak{p}_K^{N+i}\Gamma}{1 + \mathfrak{p}_K^{N+i+1}\Gamma} \to \frac{1 + \mathfrak{p}_K^{N}\Gamma}{1 + \mathfrak{p}_K^{N+i+1}\Gamma} \to \frac{1 + \mathfrak{p}_K^{N}\Gamma}{1 + \mathfrak{p}_K^{N+i}\Gamma} \to 0,$$

so (8.2) follows by induction on i and Lemma 8.3.

It remains to show that $\hat{H}^j(1 + \mathfrak{p}^N\Gamma) = 0$. In a perfect world, we would have

$$\hat{H}^j(G, \varprojlim_n M_n) = \varprojlim_n \hat{H}^j(G, M_n)$$

for any sequence of modules with a G-action and G-equivariant structure maps. Thus would then imply

$$\begin{aligned} \hat{H}^j(1 + \mathfrak{p}_K^N\Gamma) &= \hat{H}^j\big(\varprojlim_{i\geq 0}(1 + \mathfrak{p}_K^N\Gamma)/(1 + \mathfrak{p}_K^{N+i}\Gamma)\big) \\ &= \varprojlim_{i\geq 0} \hat{H}^j((1 + \mathfrak{p}_k^N\Gamma)/(1 + \mathfrak{p}_K^{N+i}\Gamma)) \\ &= \varprojlim_{i\geq 0} 0 \\ &= 0 \end{aligned}$$

by (8.2), as our filtration is complete. Thus, we need to find some way to justify commuting Tate cohomologies and inverse limits.

LEMMA 8.4. *Suppose we have a sequence of modules*

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{n+1} & \longrightarrow & E_{n+1} & \longrightarrow & N_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_n & \longrightarrow & E_n & \longrightarrow & N_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{8.3}$$

with exact rows. Then we have an exact sequence

$$0 \to \varprojlim_n M_n \xrightarrow{\varphi} \varprojlim_n E_n \xrightarrow{\psi} \varprojlim_n N_n.$$

Moreover, if $M_n \to M_{n+1}$ is surjective for each n, then ψ is as well (otherwise, it may not be!).

Proof. Evidently, φ is injective, as if $x \in \mathrm{Ker}(\varphi)$, then each coordinate of its image is 0, so by compatibility and injectivity of $M_n \to E_n$ for each n, each coordinate of x is 0, hence $x = 0$. Similarly, $\mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$ by exactness of each row in (8.3).

To see (intuitively) how surjectivity of ψ can fail, consider a compatible system $(x_n) \in \varprojlim_n N_n$. We can lift each x_n to some $y_n \in E_n$, but it is unclear how to do it compatibly, so that $(y_n) \in \varprojlim_n E_n$.

Now assume that that each of the maps $M_n \to M_{n+1}$ is surjective. Let $(x_n) \in \varprojlim_n N_n$, and suppose we have constructed $y_n \in E_n$ for some n. Choose any $\tilde{y}_{n+1} \mapsto x_{n+1}$ via the map $E_{n+1} \to N_{n+1}$, and let α_{n+1} be the image of $\tilde{y}_{n+1}$ in E_n. Then $y_n - \alpha_{n+1} \in M_n$ as it vanishes in N_n, and it lifts to $\beta_{n+1} \in M_{n+1}$ by assumption. If we now define $y_{n+1} := \tilde{y}_{n+1} + \beta_{n+1}$, then this maps to $\alpha_{n+1} + y_n - \alpha_{n+1} = y_n$ in E_n and to x_{n+1} in N_{n+1} as β_{n+1} maps to 0 in N_{n+1}, hence by induction there exists a compatible system $(y_n) \in \varprojlim_n E_n$ mapping to (x_n) via ψ, as desired (note that we may express this result as a surjection $E_{n+1} \twoheadrightarrow E_n \times_{N_n} N_{n+1}$, i.e., to the fibre product).. $\square$

Proposition 8.5. *If*

$$\cdots \twoheadrightarrow M_{n+1} \twoheadrightarrow M_n \twoheadrightarrow \cdots \twoheadrightarrow M_0 \tag{8.4}$$

is a sequence of G-modules, and $\hat{H}^i(M_{n+1}) \twoheadrightarrow \hat{H}^i(M_n)$ for all n and i, then

$$\hat{H}^i(\varprojlim_n M_n) = \varprojlim_n \hat{H}^i(M_n)$$

for all i.

Proof. We provide a proof for $\hat{H}^0$. Let $M := \varprojlim_n M_n$, so that we are comparing $\hat{H}^0(M) = M^G/\mathrm{N}(M)$ and $\varprojlim_n \hat{H}^0(M_n) = \varprojlim_n M_n^G/\mathrm{N}(M_n)$. This amounts to showing that the natural map

$$(\varprojlim_n M_n^G)/(\varprojlim_n \mathrm{N}(M_n)) \xrightarrow{\sim} \varprojlim_n (M_n^G/\mathrm{N}(M_n))$$

is an isomorphism. We have a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{N}(M_{n+1}) \arrow[r] \arrow[d, "\alpha_n"] & M_{n+1}^G \arrow[r] \arrow[d] & \hat{H}^0(G, M_{n+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{N}(M_n) \arrow[r] \arrow[d] & M_n^G \arrow[r] \arrow[d] & \hat{H}^0(G, M_n) \arrow[r] \arrow[d] & 0, \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and we claim that α_n is surjective for each n. Indeed, let $x \in \mathrm{N}(M_n)$, so that $x = \mathrm{N}(y)$ for some $y \in M_n$. Lifting y to an element $z \in M_{n+1}$, we have $\alpha_n(\mathrm{N}(z)) = x$, as desired. Thus, by Lemma 8.4, we have

$$\varprojlim_n \hat{H}^0(M_n) = (\varprojlim_n M_n^G)/(\varprojlim_n \mathrm{N}(M_n)).$$

Now, we have

$$\hat{H}^0(\varprojlim_n M_n) = (\varprojlim_n M_n)^G/\mathrm{N}(\varprojlim_n M_n) = M^G/\mathrm{N}(M).$$

It is clear that $(\varprojlim_n M_n)^G = \varprojlim_n M_n^G$, since G acts on each of the coordinates of $\varprojlim_n M_n$, so it remains to show that $\mathrm{N}(\varprojlim_n M_n) \xrightarrow{\sim} \varprojlim_n \mathrm{N}(M_n)$. Letting $K_n := \mathrm{Ker}(\mathrm{N}\colon M_n \to M_n)$ for each n, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{n+1} & \longrightarrow & M_{n+1} & \longrightarrow & \mathrm{N}(M_{n+1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_n} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & M_n & \longrightarrow & \mathrm{N}(M_n) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We'd like to show that β_n is surjective. Recall that $\hat{H}^1(G, M_n) = K_n/(1-\sigma)M_n$, and thus we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (1-\sigma)M_{n+1} & \longrightarrow & K_{n+1} & \longrightarrow & \hat{H}^1(G, M_{n+1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \delta_n} & & \\
0 & \longrightarrow & (1-\sigma)M_n & \longrightarrow & K_n & \longrightarrow & \hat{H}^1(G, M_n) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Now, γ_n is surjective (the proof is similar to that for α_n), and δ_n is surjective by hypothesis. Thus, β_n is surjective by the Snake Lemma, and so Lemma 8.4 implies

$$\varprojlim_n \mathrm{N}(M_n) = (\varprojlim_n M_n)/(\varprojlim_n K_n) = \mathrm{N}(\varprojlim_n M_n).$$

It follows that $\hat{H}^0(\varprojlim_n M_n) = \varprojlim_n \hat{H}^0(M_n)$, as desired. $\square$

COROLLARY 8.6. *For a sequence (8.4), if* $\hat{H}^i(M_n) = 0$ *for all* n *and* i*, then* $\hat{H}^i(\varprojlim_n M_n) = 0$*. In particular,* $\hat{H}^i(1 + \mathfrak{p}_K^N\Gamma) = 0$*, where we have set* $M_i := (1 + \mathfrak{p}_K^N\Gamma)/(1 + \mathfrak{p}_K^{N+i}\Gamma)$ *for each* i*.*

It follows that, since $1 + \mathfrak{p}_K^N\Gamma \subseteq \mathcal{O}_L^\times$ is a (additive, with a normal basis) finite-index subgroup, we have $\chi(\mathcal{O}_L^\times) = \chi(1+\mathfrak{p}_K^N) = 1$, which establishes (2) of Claim 7.8.

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