LECTURE 6

Exact Sequences and Tate Cohomology

Last time we began discussing some simple homological algebra; our motivation was to compute the order of certain finite abelian groups (in particular, $K^\times/\mathrm{N}(L^\times)$, where L/K is a cyclic extension of local fields). Recall the following definition:

Definition 6.1. A sequence

$$\cdots \to X^{n-1} \xrightarrow{d^n} X^n \xrightarrow{d^{n+1}} X^{n+1} \to \cdots$$

is *exact* if for each n, we have $\mathrm{Ker}(d^{n+1}) = \mathrm{Im}(d^n)$, where we refer to the 'd^i' as *differentials*.

To solve this equation, one typically shows that if d^{n+1} kills an element, then it is in the image of d^n. We saw that for a short exact sequence

$$0 \to M \hookrightarrow E \twoheadrightarrow N \to 0,$$

we have $M = E/N$ and $\#E = \#M \cdot \#N$, so short exact sequences are an effective way of measuring the size of abelian groups. We also saw that for any such short exact sequence and $n \geq 1$, there is a long exact sequence

$$0 \to M[n] \to E[n] \to N[n] \xrightarrow{\delta} M/n \to E/n \to N/n \to 0, \tag{6.1}$$

where we recall that

$$M[n] := \{x \in M : nx = 0\} = \mathrm{Tor}_1(M, \mathbb{Z}/n) = H_1(M \otimes_L \mathbb{Z}/n),$$

which denote the torsion subgroup and first homology group, respectively, and similarly for E and N. The boundary map δ lifts an element $x \in N[n]$ to $\tilde{x} \in E$, so that $n\tilde{x} \in M$ since $nx = 0$ in N, and then maps $n\tilde{x}$ to its equivalence class in M/n. It remains to check the following claims:

Claim 6.2. *The boundary map δ is well-defined.*

Proof. Suppose $\tilde{\tilde{x}}$ is another lift of x. Then $\tilde{x} - \tilde{\tilde{x}} \in M$ as its image in N is zero, hence $n(\tilde{x} - \tilde{\tilde{x}}) \in nM$, so $n\tilde{x} = n\tilde{\tilde{x}}$ in M/nM. □

Claim 6.3. *The sequence in (6.1) is exact.*

Proof. This is clear at all maps aside from the boundary map. If $\delta(x) = n\tilde{x} = 0$ in M/n for some $x \in N[n]$ with lift $\tilde{x} \in E$, then $\tilde{x} \in M$, and therefore $x = 0$ in N. Hence $x \in N[n]$ and so $\tilde{x} \in E[n]$ by exactness. Similarly, if $x \in M/n$ has image zero E/n, then $\tilde{x} = ny$ for some $y \in E$, where $\tilde{x}$ is a lift of x to M. Projecting down to N, we see that $0 = n\bar{y}$ by exactness, and therefore $\bar{y} \in N[n]$. So $ny \in M$, again by exactness, and $\delta(\bar{y}) = ny = x$ as classes in M/n, as desired. □

We have the following useful lemma:

LEMMA 6.4. *Suppose*

$$0 \to X^0 \xrightarrow{d^1} X^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^{n-1} \xrightarrow{d^n} X^n \to 0$$

is exact, and all X^i are finite. Then

$$\# X^0 \cdot \# X^2 \cdots = \# X^1 \cdot \# X^3 \cdots .$$

PROOF. We proceed by induction on n. The result is clear for $n = 1$, so suppose it holds for $n - 1$. Form the exact sequences

$$0 \to X^0 \to \cdots \to X^{n-1} \xrightarrow{d^{n-1}} \operatorname{Im}(d^{n-1}) \to 0$$

and

$$0 \to \operatorname{Im}(d^{n-1}) \to X^{n-1} \xrightarrow{d^n} X^n \to 0.$$

Suppose n is even. Then

$$\begin{aligned} \# X^0 \cdot \# X^2 \cdots \# X^n &= \# X^0 \cdot \# X^2 \cdots \# X^{n-1} \cdot \frac{\# X^{n-1}}{\# \operatorname{Im}(d^{n-1})} \\ &= \# X^1 \cdot \# X^3 \cdots \# \operatorname{Im}(d^{n-1}) \cdot \frac{X^{n-1}}{\# \operatorname{Im}(d^{n-1})} \\ &= \# X^1 \cdot \# X^3 \cdots \# X^{n-1}, \end{aligned}$$

by the inductive hypothesis. The proof for odd n is similar. □

DEFINITION 6.5. Let M be an abelian group with M/n and $M[n]$ finite. Then

$$\chi(M) := \chi_n(M) := \frac{\#(M/n)}{\#(M[n])}$$

is the *Euler characteristic* of M.

EXAMPLE 6.6.
(1) If M is finite, then $\chi(M) = 1$. To see this, observe that

$$0 \to M[n] \to M \xrightarrow{n} M \to M/n \to 0$$

is exact, and so by Lemma 6.4, $\#(M[n]) \cdot \# M = \# M \cdot \#(M/n)$.

(2) If $M = \mathbb{Z}$, then $\chi(M) = n$, since $M[n] = 0$ and $M/n = \mathbb{Z}/n$ has order n.

The following lemma is an important fact about Euler characteristics:

LEMMA 6.7. *For a short exact sequence*

$$0 \to M \to E \to N \to 0,$$

if χ exists for two of the three abelian groups, then it exists for the third, and $\chi(M) \cdot \chi(N) = \chi(E)$, where "exists" means that (say for M) M/n and $M[n]$ are both finite.

PROOF. We have an exact sequence

$$0 \to M[n] \to E[n] \to N[n] \to M/n \to E/n \to N/n \to 0.$$

More generally, note that if $X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ is exact, then X^n is finite if X^{n-1} and X^{n+1} are, since there is a short exact sequence

$$0 \to \operatorname{Im}(d^{n-1}) = \operatorname{Ker}(d^n) \to X^n \to \operatorname{Im}(d^n) \to 0,$$

where the outer two groups are finite and therefore $\#X^n = \#\mathrm{Ker}(d^n) \cdot \#\,\mathrm{Im}(d^n)$ is too. Thus, all groups in the sequence are finite, and

$$\#(M[n]) \cdot \#(N[n]) \cdot \#(E/n) = \#(E[n]) \cdot \#(M/n) \cdot \#(N/n)$$

by Lemma 6.4, which yields the desired expression. □

As an application, let us compute $\#(K^\times/(K^\times)^n)$. Observe that

$$\chi(K^\times) = \frac{\#(K^\times/(K^\times)^n)}{\#(K^\times[n])},$$

where the denominator is the number of nth roots of unity in K. Moreover, we have an exact sequence

$$0 \to \mathcal{O}_K^\times \to K^\times \xrightarrow{v} \mathbb{Z} \to 0,$$

and so by Lemma 6.7, $\chi(K^\times) = \chi(\mathcal{O}_K^\times)\chi(\mathbb{Z}) = n\chi(\mathcal{O}_K^\times)$. Thus, we'd really like to compute $\chi(\mathcal{O}_K^\times)$.

A good heuristic to use is that if $\mathcal{O}_K^\times$ contains some open, that is, finite index, subgroup Γ, then $\Gamma \simeq \mathcal{O}_K^+$, which is true if $\mathrm{char}(K) = 0$ by p-adic exponentials. It then follows that

$$\chi(\mathcal{O}_K^\times) = \chi(\Gamma)\chi(\mathcal{O}_K^\times/\Gamma) = \chi(\Gamma) = \chi(\mathcal{O}_K) \tag{6.2}$$

under addition, since $\mathcal{O}_K^\times/\Gamma$ is finite by assumption. Then $\mathcal{O}_K[n] = 0$ additively (since $\mathcal{O}_K$ is an integral domain), and $\chi(\mathcal{O}_K) = \#(\mathcal{O}_K/n) = |n|_K^{-1}$, where $|x|_K := q^{-v(x)}$ denotes the normalized (i.e., $v(\pi) = 1$ for a uniformizer π) absolute value inside K, and q denotes the order of the residue field. The resulting formula

$$\#(K^\times/(K^\times)^n) = \frac{n \cdot \#(K^\times[n])}{|n|_K} \tag{6.3}$$

recovers that already proven in Problem 1(b) of Problem Set 1 for $n = 2$ (though the same methods would also work for general n). The proof without exponentials uses the fact that, for large enough N,

$$1 + \mathfrak{p}^N \xrightarrow{x \mapsto x^n} 1 + \mathfrak{p}^{N+v(n)}$$

is an isomorphism (which can be shown using filtrations; this is the multiplicative version of the additive statement we had earlier).

We now introduce the notion of Tate cohomology for cyclic groups.

Definition 6.8. If G is a (not necessarily finite) group, then a *G-module* A is an abelian group, with G acting on A by group automorphism. Equivalently, there is a homomorphism $G \to \mathrm{Aut}(A)$, where the action of G satisfies

(1) $g \cdot (a + b) = g \cdot a + g \cdot b$,
(2) $(gh) \cdot a = g \cdot (h \cdot a)$,

for all $g, h \in G$ and $a, b \in A$.

Example 6.9. If L/K is an extension of fields with $G := \mathrm{Gal}(L/K)$, then L and $L^\times$ are G-modules, since field automorphisms preserve both operations. This will be the main example concerning us.

Now, assume G is finite, and let A be a G-module.

DEFINITION 6.10. The *first Tate cohomology group* is

$$\hat{H}^0(G, A) := A^G/\mathrm{N}(A),$$

where

$$A^G := \{a \in A : g \cdot a = a \text{ for all } g \in G\}$$

denotes the set of fixed points.

Note that the norm map is defined as

$$\mathrm{N}\colon A \to A, \quad a \mapsto \sum_{g \in G} g \cdot a,$$

so we really do need the assumption that G be finite. Moreover, this expression shows that the norm map factors through $A^G \subseteq A$.

EXAMPLE 6.11.
(1) Returning to Example 6.9 with $A = L$, we have $A^G = K$, and $\mathrm{N}\colon L \to K$ is the field trace, hence $\hat{H}^0(L/K) = K/\mathrm{T}(L) = 0$, since L/K must be separable.
(2) If $A = L^\times$, then $(L^\times)^G = K^\times$, and $\hat{H}^0(L^\times) = K^\times/\mathrm{N}(L^\times)$. Thus, our earlier problem is now rephrased as computing $\hat{H}^0(G, L^\times)$ for L/K a cyclic extension of local fields.
(3) If A is any abelian group, then we say that G acts on A trivially if $g \cdot a = a$ for all $g \in G$ and $a \in A$. Then $\hat{H}^0(G, A) = A/\#G$. Thus, the notion of Tate cohomology entirely generalizes our previous discussion.

DEFINITION 6.12. A *map* (or *G-morphism*, or any other reasonable nomenclature) of G-modules $A \xrightarrow{f} B$ is a group homomorphism preserving the action of G, that is, $f(g \cdot a) = g \cdot f(a)$ for all $g \in G$ and $a \in A$.

A *(short) exact sequence of G-modules* is a (short) exact sequence of abelian groups, but where all maps are G-morphisms.

EXAMPLE 6.13. $1 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{v} \mathbb{Z} \to 1$ is a short exact sequence of G-modules, where $G := \mathrm{Gal}(L/K)$ and G acts trivially on $\mathbb{Z}$ and on $\mathcal{O}_L^\times$ via the Galois action.

Now, let

$$0 \to A \to B \to C \to 0$$

by a short exact sequence of G-modules. Then we obtain an exact sequence

$$\hat{H}^0(G, A) \xrightarrow{\alpha} \hat{H}^0(G, B) \xrightarrow{\beta} \hat{H}^0(G, C), \tag{6.4}$$

where α is not necessarily injective (as we saw when the group action was trivial in the previous lecture), and β is not necessarily surjective. This is because Tate cohomology involves two operations: one, taking fixed points, is left-exact, but not right-exact, and the other, taking a quotient, is right-exact but not left-exact.

Now, assume $G = \mathbb{Z}/n\mathbb{Z}$, and let $\sigma \in G$ be a generator (i.e. 1).

DEFINITION 6.14. The *second Tate cohomology group* is

$$\hat{H}^1(G, A) := \mathrm{Ker}(\mathrm{N}\colon A \to A)/(1 - \sigma)A.$$

Note that the reason we take the quotient is because, for any $x := y - \sigma y$ for $y \in A$, we get

$$\mathrm{N}(x) = x + \sigma x + \cdots + \sigma^{n-1}x = y - \sigma y + \sigma y - \sigma^2 y + \cdots + \sigma^{n-1}y - \underbrace{\sigma^n y}_{y} = 0,$$

and we'd like to omit these trivial cases for the kernel.

Now, we claim that for an exact sequence

$$0 \to A \to B \to C \to 0,$$

there is an exact sequence

$$\hat{H}^0(A) \to \hat{H}^0(B) \to \hat{H}^0(C) \xrightarrow{\delta} \hat{H}^1(A) \to \hat{H}^1(B) \to \hat{H}^1(C) \tag{6.5}$$

via the boundary map δ, which lifts any $x \in C^G/\mathrm{N}(C)$ to $\tilde{x} \in B$, and then takes $(1-\sigma)\tilde{x}$. Since $x \in C^G$, we have $(1-\sigma)x = 0$ in C, and therefore $(1-\sigma)\tilde{x} \in A$. Moreover, $(1-\sigma)\tilde{x}$ is clearly killed by the norm in A, hence it gives a class in $\hat{H}^1(G, A)$. Again, we check the following:

Claim 6.15. *The boundary map δ is well-defined, i.e., it doesn't depend on the choice of $\tilde{x}$.*

Proof. If $\tilde{\tilde{x}}$ is another lift, then $\tilde{x} - \tilde{\tilde{x}} \in A$ since $C \simeq B/A$, so $(1-\sigma)(\tilde{x} - \tilde{\tilde{x}})$ is zero in $\hat{H}^1(G, A)$. □

Claim 6.16. *The sequence (6.5) extends to be exact.*

Proof. As before, we verify this only at the boundary map. Letting $x \in B^G/\mathrm{N}(B)$, its image in $\hat{H}^1(A)$ is $(1-\sigma)x = 0$. If $x \in \mathrm{Ker}(\delta)$, then $\tilde{x} \in B^G$ and hence in $\hat{H}^0(B)$ for some lift $\tilde{x}$ of x.

Letting $x \in C^G/\mathrm{N}(C)$, its image in $\hat{H}^1(A)$ is $(1-\sigma)\tilde{x}$, where $\tilde{x}$ is a lift of x to B, hence it is killed in $\hat{H}^1(B)$ by definition. If $x \in \hat{H}^1(A)$ is 0 in $\hat{H}^1(B)$, then $x \in (1-\sigma)B$, hence $x \in \mathrm{Im}(\delta)$. □