LECTURE 19

# Brauer Groups

In this lecture, we present an overview of Brauer groups. Our presentation will be short on proofs, but we will give precise constructions and formulations of claims. For complete proofs, see [**Mil13, Ser79, Boy07**]. Our motivating question is: "what was all that stuff about Hamiltonian algebras?" (see Problem 5 of Problem Set 1, Problem 3 of Problem Set 2, and Problem 2 of Problem Set 3). We will see that there are two objects called the "Brauer group," one which has a cohomological definition, and one which has a more general algebraic definition; we'll show that the two coincide.

Recall that, if $L/K$ is a $G$-Galois extension of nonarchimedean local fields, then $\mathbb{Z}[-2]^{\mathrm{t}G} \simeq (L^\times)^{\mathrm{t}G}$. When we took $H^0$ (zeroth cohomology), we obtained LCFT, that is, $K^\times/\mathrm{N}L^\times \simeq H_1(G,\mathbb{Z}) = G^{\mathrm{ab}}$. This is natural, as $L^\times$ is in degree 0. But $\mathbb{Z}$ is in degree 2, so what if we take $H^2$? Well, we obtain an isomorphism

$$\hat{H}^0(G, Z) \simeq \hat{H}^2(G, L^\times),$$

where the left-hand side is very simply isomorphic to

$$\mathbb{Z}/[L:K]\mathbb{Z} = \mathbb{Z}/\#G\mathbb{Z} = \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z},$$

since the invariants are $\mathbb{Z}$ as $G$ acts trivially on $\mathbb{Z}$, and the norms correspond to multiplication by $\#G$. However, the right-hand side is more mysterious, motivating the following definition:

Definition 19.1. The *cohomological Brauer group of $L/K$* is

$$\mathrm{Br}^{\mathrm{coh}}(K/L) := H^2(G, L^\times).$$

Remark 19.2. What happens if we vary $L$? Suppose we have Galois extensions $L_2/L_1/K$, with $\mathrm{Gal}(L_i/K) = G_i$ for $i = 1, 2$. Then we have a short exact sequence

$$0 \to \mathrm{Gal}(L_2/L_1) \to G_2 \to G_1 \to 0,$$

and maps

$$\mathrm{Br}^{\mathrm{coh}}(L_1/K) = H^2(G_1, L_1^\times) \to H^2(G_2, L_1^\times) \to H^2(G_2, L_2^\times) = \mathrm{Br}^{\mathrm{coh}}(L_2/K)$$

since invariance with respect to $G_2$ implies impvariance with respect to $G_1$, and via the embedding $L_1^\times \hookrightarrow L_2^\times$. This motivates the following definition.

Definition 19.3. The *cohomological Brauer group of $K$* is

$$\mathrm{Br}^{\mathrm{coh}}(K) := \varinjlim_{L/K} \mathrm{Br}^{\mathrm{coh}}(K/L) = H^2(\mathrm{Gal}(K), \overline{K}^\times),$$

where the directed limit is over finite Galois extensions $L/K$.

Note that the right-most expression above uses our notation from last lecture.

Claim 19.4. *Under* LCFT*, the following diagram commutes:*

$$\begin{tikzcd} H^2(G_2, L_1^\times) \arrow[r] \arrow[d, equal] & H^2(G_1, L_1^\times) \arrow[d, equal] \\ \frac{1}{[L_1:K]}\mathbb{Z}/\mathbb{Z} \arrow[r, hook] & \frac{1}{[L_2:K]}\mathbb{Z}/\mathbb{Z}. \end{tikzcd}$$

Corollary 19.5. *For a nonarchimedean local field $K$, we have*

$$\mathrm{Br}^{\mathrm{coh}}(K) \simeq \mathbb{Q}/\mathbb{Z}.$$

Remark 19.6. One can also show that the top-most map is injective. For an extension $L/K$ of nonarchimedean local fields, there is an exact sequence

$$0 \to \mathrm{Br}^{\mathrm{coh}}(K/L) \to \underbrace{\mathrm{Br}^{\mathrm{coh}}(K)}_{H^2(\mathrm{Gal}(K), \overline{K}^\times)} \to \underbrace{\mathrm{Br}^{\mathrm{coh}}(L)}_{H^2(\mathrm{Gal}(L), \overline{K}^\times)},$$

which we'll justify next time.

We now turn to the algebraic perspective, which provides the classical definition of the Brauer group.

Proposition 19.7. *Let $K$ be a field, and let $A$ be a finite-dimensional $K$-algebra with center $K$. Then the following are equivalent:*

*(1) $A$ is simple, that is, it has no non-trivial 2-sided ideals.*
*(2) $A \otimes_K L \simeq \mathrm{M}_n(L)$ (i.e., an $n \times n$ matrix algebra) for some separable extension $L/K$.*
*(3) $A \simeq \mathrm{M}_n(D)$ for some central (i.e., with center $K$) division (i.e., multiplicative inverses exist, but multiplication is not necessarily commutative) algebra $D$ over $K$.*

*If these conditions hold, then $A$ is called a* central simple algebra *(*CSA*; alternatively,* Azumaya algebra*) over $K$.*

Corollary 19.8. *Any central simple algebra over $K$ has dimension a square.*

Proof. The dimension of $A$ is preserved by the tensoring operation in (2), and $\mathrm{M}_n(L)$ has square dimension. □

Example 19.9. (1) $\mathrm{M}_n(K)$.
(2) A central division algebra over $K$.
(3) The Hamiltonians over $R := K$.
(4) For all fields $K$ with $\mathrm{char}(K) \neq 2$ and elements $a, b \in K^\times$, $H_{a,b}$ is a CSA.

However, a general field $L/K$ does not satisfy the centrality property, and indeed, its dimension might not be a square.

Definition 19.10. The CSA *Brauer group of $K$* is

$$\mathrm{Br}^{\mathrm{csa}}(K) := \{\text{equivalence classes of CSAs over } K\}$$

with respect to the equivalence relation $A \simeq B$ if and only if $A$ and $B$ are matrix algebras over isomorphic division algebras in (3) above (not necessarily of the same dimension).

Remark 19.11. The isomorphism class of $A$ depends only on the isomorphism class of $D$.

Proposition 19.12. *If $A$ and $B$ are* CSA*s over $K$, then $A \otimes_K B$ is also a* CSA *(note that tensor products multiply dimension, so $A \otimes_K B$ also has square dimension). Up to equivalence, this only depends on $[A], [B] \in \mathrm{Br}^{\mathrm{csa}}(K)$, so $\mathrm{Br}^{\mathrm{csa}}(K)$ forms an abelian group.*

The proof is omitted; showing that $A \otimes_K B$ is simple is rather annoying. We note, however, that matrix algebras over $K$ represent the identity element (or "0 equivalence class") of $\mathrm{Br}^{\mathrm{csa}}(K)$. The inverse of $A$ is $A$, but with opposite multiplication (i.e., $x \cdot y := yx$), which we denote $A^{\mathrm{op}}$. Indeed, we have a canonical algebra homomorphism

$$A \otimes_K A^{\mathrm{op}} \to \mathrm{End}_K(A),$$

with $A$ and $A^{\mathrm{op}}$ acting on opposite sides (note that $\mathrm{End}_K(A)$ is in the 0 equivalence class of $\mathrm{Br}^{\mathrm{csa}}(K)$). The kernel of this (nonzero) map must be a 2-sided ideal, so since $A \otimes_K A^{\mathrm{op}}$ is simple, it must be injective. Since both sides have dimension $\dim_K(A)^2$ over $K$, it is therefore also surjective, hence an isomorphism.

Definition 19.13. The CSA *Brauer group of $L/K$* is

$$\mathrm{Br}^{\mathrm{csa}}(K/L) := \{\text{equivalence classes of CSAs } A : A \otimes_K L \simeq \mathrm{M}_n(L)\},$$

and we say that such an $A$ *splits* over $L$.

This notion is equivalent to the underlying division algebra $D$ splitting over $L$, which likewise means that $D \otimes_K L \simeq \mathrm{M}_n(L)$ is no longer a division algebra. Then clearly

$$\mathrm{Br}^{\mathrm{csa}}(K) = \bigcup_{\substack{L/K \\ \text{seperable}}} \mathrm{Br}^{\mathrm{csa}}(K/L).$$

Example 19.14.

(1) If $K$ is algebraically closed, then $\mathrm{Br}^{\mathrm{csa}}(K) = \{0\}$ is clearly trivial.
(2) $\mathrm{Br}^{\mathrm{csa}}(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\} = \mathbb{Z}/2\mathbb{Z}$, since the only other division algebra over $R$ is $\mathbb{C}$ which is not central. Note that the Hamiltonians $\mathbb{H}$ split over $\mathbb{C}$ and have dimension $4 = 2^2$.
(3) $H_{a,b}$ splits over $K(\sqrt{a})$ and $K(\sqrt{b})$, which are the usual commutative subalgebras of $H_{a,b}$. In fact, we can take $K(\sqrt{c})$ for any $c \in K$, and can also conjugate by units. Thus, in the following claim, $L$ is very non-unique. Note that, unlike in this case, such an $L$ need not be Galois over $K$; examples are hard to find, but were discovered in the 1970s.

Claim 19.15. *An $n^2$-dimensional central division algebra $D/K$ splits over a degree-$n$ extension $L/K$ if and only if $K \subseteq L \subseteq D$ as a subalgebra. Equivalently, $L$ is a maximal commutative subalgebra in $D$.*

First, we have the (easy) fact that any commutative subalgebra of a division algebra is a field. This is similar to the fact that every finite-dimensional integral domain over a field is itself a field. This is because multiplication by any element is injective, and given by a matrix, hence surjective. Thus, some element maps to the identity, providing the desired inverse element. Then there is the (non-obvious) fact that $\dim_K(D)$ is the square of the dimension of any maximal commutative subalgebra of $D$ over $K$. We prove one direction of the claim:

PROOF. Suppose $K \subseteq L \subseteq D$, where $[L : K] = n$ and $\dim_K(D) = n^2$. We claim that $D$ splits over $L$. We have a map

$$D \otimes_K L \to \mathrm{End}_L(D) = \mathrm{M}_n(L)$$

with $L$ acting on $D$ via right multiplication and $D$ acting on itself via left multiplication; the two actions commute. This map is injective (if $dxl = x$ for some $d \in D$, $l \in L$, and all $x \in D$, then $dl = 1$, hence $dx = xl^{-1} = xd$ for all $x \in D$ and therefore $d \in K$, so $d \otimes l = 1 \otimes 1$), and therefore surjective and an isomorphism, as desired. $\square$

THEOREM 19.16. *For every Galois extension $L/K$, we have $\mathrm{Br}^{\mathrm{coh}}(K) \simeq \mathrm{Br}^{\mathrm{csa}}(K)$ and $\mathrm{Br}^{\mathrm{coh}}(K/L) \simeq \mathrm{Br}^{\mathrm{csa}}(K/L)$.*

Note that the cohomological Brauer group is defined only for Galois extensions $L/K$, whereas the CSA Brauer group is defined for all extensions $L/K$. This gives meaning to the cohomological definition of the Brauer group. We provide the following (incomplete) proof sketch:

PROOF. For a $G$-Galois extension $L/K$, we define a map

$$H^2(G, L^\times) \to \mathrm{Br}^{\mathrm{csa}}(K/L). \tag{19.1}$$

Every element in $H^2$ is represented by a 2-cocycle, that is, a map $\varphi\colon G \times G \to L^\times$ satisfying

$$^{g_1}_{\varphi(g_2,g_3)}\varphi(g_1, g_2g_3) = \varphi(g_1g_2, g_3)\varphi(g_1, g_2)$$

for every $g_1, g_2, g_3 \in G$, and where we have introduced the notation $^g x := g \cdot x$. We define a CSA associated to each such $\varphi$ as follows: form

$$L[G] = \left\{ \textstyle\sum_{g \in G} x_g[g] : x_g \in L \right\}$$

subject to the relations

$$[g]x = {}^g x[g] \quad \text{and} \quad [g][h] = \varphi(g, h)[gh]$$

for each $x \in L$ and $g, h \in G$. We now check that the 2-cocyle identity is equivalent to associativity:

$$\begin{aligned}
[g_1]([g_2][g_3]) &= [g_1]\varphi(g_2, g_3)[g_2g_3] \\
&= {}^{g_1}\varphi(g_2, g_3)[g_1][g_2g_3] \\
&= {}^{g_1}\varphi(g_2, g_3)\varphi(g_1, g_2g_3)[g_1g_2g_3] \\
&= \varphi(g_1g_2, g_3)\varphi(g_1, g_2)[g_1g_2g_3] \\
&= \varphi(g_1, g_2)[g_1g_2][g_3] \\
&= ([g_1][g_2])[g_3],
\end{aligned}$$

for all $g_1, g_2, g_3 \in G$. We claim, but do not prove, that the equivalence class of this CSA only depends on $\varphi$, up to coboundaries, and that it splits over $L$. Moreover, our map (19.1) is a group isomorphism. $\square$

This is not an especially deep theorem, despite being far from obvious; the complete proof uses a lot of structure theory that is not particularly memorable.

We now ask, how many isomorphism classes of central division algebras over $K$ of degree $n^2$ are there? When $n = 1^2$, there is only 1; when $n = 2^2$, there is again only 1, as shown in Problem 2(d) of Problem Set 3.

CLAIM 19.17. *There are $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$ isomorphism classes of central division algebras over $K$ of degree $n^2$.*

CLAIM 19.18. *Let $L/K$ be a degree-$n$ extension of nonarchimedean local fields. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Br}(K) & \longrightarrow & \mathrm{Br}(L) \\ \Big\| \scriptstyle\text{LCFT} & & \Big\| \scriptstyle\text{LCFT} \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{\times n} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

Note that we have simply denoted the Brauer group by Br in light of Theorem 19.16. Indeed, we've seen that both horizontal maps should have non-trivial kernel; when $L = \overline{K}$, $\mathrm{Br}(L)$ is trivial. Then a central division algebra of degree $n^2$ has order $n$ in $\mathrm{Br}(K)$, i.e., is $n$-torsion, since it splits over its degree-$n$ maximal commutative subalgebra. Alternatively, $H^2(G, -)$ is $\#G$-torsion, as proven in Problem 2(c) of Problem Set 7.

COROLLARY 19.19. *Any degree-$n^2$ division algebra splits over any degree-$n$ extension of $K$.*

On the other hand, if $A \simeq \mathrm{M}_n(D)$ is a CSA over $K$ of degree $n^2$, representing some $n$-torsion class in $\mathrm{Br}(K)$, then $A$ splits over the maximal commutative subalgebra of $D$. This implies that $A$ is itself a division algebra if it is $n$-torsion, but not $m$-torsion for any $m \mid n$. This gives $\varphi(n)$: the division algebras come from classes in $\mathbb{Z}/n\mathbb{Z}$ which do not arise from any smaller $\mathbb{Z}/m\mathbb{Z}$. Another upshot is the following:

COROLLARY 19.20. *Any degree-$n^2$ central division algebra over $K$ contains every degree-$n$ field extension of $L$.*

PROOF. Any $n$-torsion class in $\mathrm{Br}(K)$ maps to zero in the Brauer group $\mathrm{Br}(L)$ over any degree-$n$ extension $L$ of $K$ by Claim 19.18. Thus, it splits over $L$, hence contains it by Claim 19.15. $\square$

For $n = 2$, this is the result that every $x \in K$ admits a square in $H_{a,b}$, which was shown in Problem 2(c) of Problem Set 3.