LECTURE 11

The Mapping Complex and Projective Resolutions

Throughout, A will be an associative algebra (which might not be commutative), e.g. $A = \mathbb{Z}, \mathbb{Z}[G]$, where G is a (usually finite) group. Recall that we wanted rules by which $X \mapsto X^{\mathrm{h}G}, X_{\mathrm{h}G}$, where X is a complex of G-modules and $X^{\mathrm{h}G}$ and $X_{\mathrm{h}G}$ are complexes of abelian groups. Our guiding "axioms" for this construction will be:

(1) If X is *acyclic*, i.e., $H^i(X) = 0$ for all $i \in \mathbb{Z}$, then we'd like $X_{\mathrm{h}G}$ and $X^{\mathrm{h}G}$ to be acyclic also.
(2) Both $X^{\mathrm{h}G}$ and $X_{\mathrm{h}G}$ should commute with cones, i.e., if $f \colon X \to Y$ is a map of complexes of G-modules, then $\mathrm{hCoker}(f)^{\mathrm{h}G} \simeq \mathrm{hCoker}(X^{\mathrm{h}G} \to Y^{\mathrm{h}G})$. This condition is relatively simple to satisfy, as it merely amounts to commuting with finite direct sums and shifts by the proof of Claim 10.1.
(3) The construction should have something to do with invariants and coinvariants. Namely, if $X = (\cdots \to 0 \to M \to 0 \to \cdots)$ is in degree 0 only, then $H^0(X^{\mathrm{h}G}) = M^G$ and
$$H^0(X_{\mathrm{h}G}) = M_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M = M \Big/ \sum_{g\in G} (g-1)M.$$

A naive and incorrect attempt would be to define
$$X^{\mathrm{h}G} := (\cdots \to (X^{-1})^G \xrightarrow{d} (X^0)^G \xrightarrow{d} (X^1)^G \to \cdots),$$
for a chain complex
$$X := (\cdots \to X^{-1} \xrightarrow{d} X^0 \xrightarrow{d} X^1 \to \cdots).$$

This trivially satisfies (2) and (3), and note that it is well-defined as the differentials commute with the group automorphisms. A weak version of (1) is satisfied: if $X \simeq 0$, i.e., the zero complex, then $X^{\mathrm{h}G}$ and $X_{\mathrm{h}G}$ are also homotopy equivalent to 0. However, this construction doesn't preserve acyclic complexes. Explicitly, if $G := \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ via multiplication by 1 and -1, then we have
$$(\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \to \cdots)^{\mathrm{h}G} = (\cdots \to 0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \cdots)$$
which is not acyclic!

A more hands-off approach is to note that if this construction preserved acyclic complexes, then since the cone of any map of acyclic complexes must be acyclic by the construction in (10.1), and since it commutes with cones by assumption, it would also preserve quasi-isomorphisms by Corollary 10.11. But we saw in Claim 10.12 that for an injection $M \overset{i}{\hookrightarrow} N$ of G-modules, we have $\mathrm{hCoker}(i) \xrightarrow{\text{qis}} \mathrm{Coker}(i) = N/M$ (where henceforth "qis" denotes a quasi-isomorphism). Thus, if the naive invariants preserved acyclic complexes, then it would also preserve cokernels, which we know to be false.

Observe that, for A an associative algebra, if $A = \mathbb{Z}[G]$, then $M \mapsto M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$, where G is given the trivial G-action. Thus, we have a general class of problems for every associative algebra A and A-module M.

Definition 11.1. Let X and Y be complexes of A-modules. Then the *mapping complex* $\underline{\mathrm{Hom}}_A(X,Y)$ is the complex of abelian groups defined by $\underline{\mathrm{Hom}}^i_A(X,Y) := \prod_{j\in\mathbb{Z}} \mathrm{Hom}_A(X^j, Y^{j+i})$, with differential $d^i f := df - (-1)^i f d$ (the signs alternate to ensure that the differential squares to zero). We can visualize this as follows:

$$
\begin{array}{ccccccc}
 & \overbrace{\cdots : \cdots \to \cdots} & & \overbrace{\cdots : \cdots \to \cdots} & & \overbrace{\cdots : \cdots \to \cdots} & \\
\to & h^{-1}\colon X^{-1} \to Y^{-2} & \xrightarrow{dh^{-1}+h^0 d} & f^{-1}\colon X^{-1} \to Y^{-1} & \xrightarrow{df^{-1}-f^0 d} & g^{-1}\colon X^{-1} \to Y^{0} & \to \\
\to & h^{0}\colon X^{0} \to Y^{-1} & \xrightarrow{dh^{0}+h^1 d} & f^{0}\colon X^{0} \to Y^{0} & \xrightarrow{df^{0}-f^1 d} & g^{0}\colon X^{0} \to Y^{1} & \to \\
\to & h^{1}\colon X^{1} \to Y^{0} & \xrightarrow{dh^{1}+h^2 d} & f^{1}\colon X^{1} \to Y^{1} & \xrightarrow{df^{1}-f^2 d} & g^{1}\colon X^{1} \to Y^{2} & \to \\
 & \underbrace{\cdots : \cdots \to \cdots} & & \underbrace{\cdots : \cdots \to \cdots} & & \underbrace{\cdots : \cdots \to \cdots} & \\
 & \pitchfork & & \pitchfork & & \pitchfork & \\
\to & \prod_{i\in\mathbb{Z}} \mathrm{Hom}(X^i, Y^{i-1}) & \xrightarrow{h\mapsto dh+hd} & \prod_{i\in\mathbb{Z}} \mathrm{Hom}(X^i, Y^{i}) & \xrightarrow{f\mapsto df-fd} & \prod_{i\in\mathbb{Z}} \mathrm{Hom}(X^i, Y^{i+1}) & \to \\
 & \| & & \| & & \| & \\
 & \underline{\mathrm{Hom}}^{-1}(X,Y) & & \underline{\mathrm{Hom}}^{0}(X,Y) & & \underline{\mathrm{Hom}}^{1}(X,Y) &
\end{array}
$$

where d denotes the respective differentials for X and Y.

Claim 11.2. *For any complexes X and Y of A-modules, $\underline{\mathrm{Hom}}_A(X,Y)$ is a complex .*

Note that a map of complexes $f\colon X \to Y$ is equivalent to an element $f = (f^i) \in \underline{\mathrm{Hom}}^0_A(X,Y)$ such that $df = 0$, where d denotes the differential on $\underline{\mathrm{Hom}}(X,Y)$. A null-homotopy of f is likewise equivalent to an element $h \in \underline{\mathrm{Hom}}^{-1}_A(X,Y)$ such that $dh = f$, with d as before. Thus, $H^0\underline{\mathrm{Hom}}(X,Y)$ is equivalent to the equivalence classes of maps $X \to Y$ modulo homotopy. This construction therefore generalizes many important notions in homological algebra.

Example 11.3. If $X := (\cdots \to 0 \to A \to 0 \to \cdots)$, with A in degree 0, then $\underline{\mathrm{Hom}}_A(X,Y) = Y$. Thus, X is what we will call projective.

Definition 11.4. A complex P of A-modules is *projective* (or *homotopy projective*, or *K-projective*, etc.; the notion was defined by Spaltenstein) if for every acyclic complex Y of A-modules, $\underline{\mathrm{Hom}}_A(P,Y)$ is also acyclic.

The issue above is that $\mathbb{Z}$ is not projective as a complex of $\mathbb{Z}[G]$-modules. We will show that we can in some sense replace $\mathbb{Z}$ "uniquely" by a projective module.

Lemma 11.5. *If P is a complex of A-modules with $P^i = 0$ for all $i \gg 0$ (i.e., the nonzero elements of P are bounded above in index), and P^i is projective as an A-module for all i, then P is projective (as a complex).*

We recall the following definition:

Definition 11.6. An A-module P^i is *projective* as an A-module if any of the following equivalent conditions hold:

(1) $\mathrm{Hom}_A(P^i, -)$ preserves surjections;
(2) $\mathrm{Hom}_A(P^i, -)$ is an exact functor;

(3) P^i is a direct summand of a free module;
(4) Given any surjection $N \twoheadrightarrow M$, every map $P^i \to M$ of A-modules lifts to a map $P^i \to N$ such that the following diagram commutes:

$$\begin{tikzcd} & N \arrow[d, two heads] \\ P^i \arrow[r] \arrow[ur, dashed] & M \arrow[d] \\ & 0. \end{tikzcd}$$

We briefly justify these equivalences. Evidently (1) and (4) are equivalent, as (4) states that if $N \twoheadrightarrow M$ then $\operatorname{Hom}_A(P^i, N) \twoheadrightarrow \operatorname{Hom}_A(P^i, M)$. Condition (2) is trivially equivalent to (1). To show (3), take $M := P^i$ and N to be some free module surjecting onto P^i (for instance, take all elements of P^i as a basis, and then just send corresponding elements to each other). Then (4) gives a splitting of $N \twoheadrightarrow P^i$, realizing P^i as a direct summand of M. It's easy to see that direct summands of projective modules are projective, so to show the converse, we simply note that free modules are projective.

Claim 11.7. *An A-module P is projective as an A-module if and only if it is projective as a complex in degree* 0.

Proof. If P is projective as a complex in degree 0, then let $f\colon N \twoheadrightarrow M$ be a surjection, and form the acyclic complex

$$X := (\cdots \to 0 \to \operatorname{Ker}(f) \to N \to M \to 0 \to \cdots).$$

Then $\underline{\operatorname{Hom}}_A(P, X)$ is

$$\cdots \to 0 \to \operatorname{Hom}_A(P, \operatorname{Ker}(f)) \to \operatorname{Hom}_A(P, N) \to \operatorname{Hom}_A(P, M) \to 0 \to \cdots,$$

and so $\operatorname{Hom}_A(P^i, -)$ preserves surjections and P is projective as an A-module by definition.

Conversely, if $X := (\cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots)$ is an acyclic complex and P is projective as an A-module, then

$$\underline{\operatorname{Hom}}_A(P, X) = (\cdots \to \operatorname{Hom}_A(P, X^{-1}) \to \operatorname{Hom}_A(P, X^0) \to \operatorname{Hom}_A(P, X^1) \to \cdots),$$

which is acyclic as if $X^{i-1} \twoheadrightarrow \operatorname{Ker}(X^i \to X^{i+1})$, then

$$\operatorname{Hom}(P, X^{i-1}) \twoheadrightarrow \operatorname{Hom}(P, \operatorname{Ker}(X^i \to X^{i+1})) = \operatorname{Ker}(\operatorname{Hom}(P, X^i) \to \operatorname{Hom}(P, X^{i+1}))$$

as $\operatorname{Hom}(P, -)$ is exact and so preserves kernels by assumption. Thus $\underline{\operatorname{Hom}}_A(P, X)$ is also acyclic and P is projective as a complex in degree 0, as desired. □

Proof (of Lemma). Let Y be an acyclic complex of A-modules. We need the following claim:

Claim 11.8. *Every map $P \to Y$ is null-homotopic.*

Proof. Let $f\colon P \to Y$. We construct a null-homotopy h of f by descending induction. For the base case, note that for all $i \gg 0$ (where this has the meaning in the statement of the lemma), we have $P^i = 0$, so $f^i = 0$, and therefore we may take $h^i := 0$. Now suppose we have maps $h\colon P^j \to Y^{j-1}$ for all $j > i$, so that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & P^{i-1} \arrow[r, "d"] \arrow[d, "f"] & P^i \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, "?"'] & P^{i+1} \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, "h"'] & \cdots \arrow[dl, "h"'] \\
\cdots \arrow[r, "d"] & Y^{i-1} \arrow[r, "d"] & Y^i \arrow[r, "d"] & Y^{i+1} \arrow[r, "d"] & \cdots .
\end{tikzcd}$$

We'd like to construct a map $h\colon P^i \to Y^{i-1}$ such that $dh + hd = f$. Observe that, for all $x \in P^i$, we have

$$(d(f - hd))(x) = ((df - (f - hd)d)(x) = (df - fd)(x) = 0$$

by the inductive hypothesis. Since $(f - hd)\colon P^i \to \operatorname{Ker}(d\colon Y^i \to Y^{i+1})$ by the previous assertion and there is a surjection $d\colon Y^{i-1} \to \operatorname{Ker}(d\colon Y^i \to Y^{i+1})$, the map $f - hd$ lifts to a map $h\colon P^i \to Y^{i-1}$ such that $dh = f - hd$ as P^i is projective by assumption. Thus, $dh + hd = f$ and h defines a null-homotopy of f, as desired. $\square$

By the claim, $H^0(\underline{\operatorname{Hom}}_A(P, Y)) = \{P \to Y\} = 0$ and $H^i(\underline{\operatorname{Hom}}_A(P, Y)) = H^0(\underline{\operatorname{Hom}}_A(P, Y[i])) = 0$ as $Y[i]$ is also acyclic for each i. Thus,the cohomologies vanish for each i and $\underline{\operatorname{Hom}}_A(P, Y)$ is therefore acyclic, so Y is projective as desired. $\square$

Our plan, approximately, will be to show that every X is quasi-isomorphic to a projective complex P, that is, $P \xrightarrow{\text{qis}} X$, called a *projective resolution* of X. Moreover, P will be "unique" or "derived" in a sense to be defined later on. Then we get some "corrected" version called $\underline{\operatorname{Hom}}_A^{\text{der}}(X, Y) := \underline{\operatorname{Hom}}_A(P, Y)$. Letting $A := \mathbb{Z}[G]$ and choosing some projective resolution $P \xrightarrow{\text{qis}} \mathbb{Z}$ (which will be very canonical, and even simpler for finite groups, though not exactly unique, although it will not matter for cohomology), we can define $X^{\text{h}G} := \underline{\operatorname{Hom}}_G(P, X)$. This will satisfy all of our axioms, as it has something to do with invariants since P is akin to $\mathbb{Z}$ and preserves acyclic complexes as P is projective!

The following proposition is sufficient to show the first point, as the complex we are interested in is $\mathbb{Z}$ in degree 0, which is trivially bounded above.

Proposition 11.9. *Let X be a complex of A-modules, and suppose X is bounded above, that is, $X^i = 0$ for all $i \gg 0$ as before. Then there exists a projective resolution $P \xrightarrow{\text{qis}} X$.*

Proof. Without loss of generality, we may assume that X is bounded above at index 0. Let P^0 be a free module surjecting onto X^0 via a map α^0 (one exists as before; simply take generators, so that the kernel consists of the relations among the generators). Then take P^{-1} to be a free module surjecting onto $P^0 \times_{X^0} X^{-1}$ as before (i.e., the fibre product over X^0):

$$\begin{tikzcd}
P^{-1} \arrow[r] \arrow[d, dashed] & P^0 \arrow[r] \arrow[d, two heads, "\alpha^0"] & 0 \arrow[r] \arrow[d] & \cdots \\
X^{-1} \arrow[r] & X^0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

This construction preserves cohomology, as $H^0X = X^0/\operatorname{Im}(X^{-1}) = P^0/P^{-1} = H^0P$, since P^{-1} surjects onto $\operatorname{Ker}(\alpha^0)$ and has image in $P^0/\operatorname{Ker}(\alpha^0) \simeq X^0$ equal to X^{-1} (as $P^0 \twoheadrightarrow X^0$). Since $P^{-1} \twoheadrightarrow X^{-1}$, we may iterate this process to construct a projective resolution P of X, as desired. $\square$

The second claim was about uniqueness of the projective resolution, which is given by the following lemma:

LEMMA 11.10. *Suppose that P_1 and P_2 are projective resolutions of a complex X of A-modules. Then there exists a homotopy equivalence γ such that the following diagram commutes up to homotopy, that is, $\beta\gamma \simeq \alpha$:*

$$\begin{tikzcd} P_1 \arrow[rr, "\gamma"] \arrow[dr, "\alpha", "\mathrm{qis}"'] & & P_2 \arrow[dl, "\beta"', "\mathrm{qis}"] \\ & X. & \end{tikzcd}$$

PROOF. Consider the following diagram:

$$\begin{tikzcd} P_1 \arrow[r, dashed, "\gamma"] \arrow[dr, "\alpha"] \arrow[ddr, "\delta"'] & P_2 \arrow[d, "\beta"] \\ & X \arrow[d] \\ & \mathrm{hCoker}(\beta). \end{tikzcd}$$

Since β is a quasi-isomorphism by assumption, $\mathrm{hCoker}(\beta)$ is acyclic by Corollary 10.11. By Claim 11.8, the composition δ is null-homotopic, hence by Claim 10.8, there is a canonical map

$$P_1 \xrightarrow{\gamma} \mathrm{hKer}(X \to \mathrm{hCoker}(\beta)) \simeq P_2$$

via homotopy equivalence, as desired. By symmetry, such a map exists in the opposite direction, hence γ is a homotopy equivalence and the diagram trivially commutes up to homotopy. $\square$

We can now ask how unique γ is here. The answer is given by the following:

CLAIM 11.11. *All such γ are homotopic.*

PROOF. We imitate the proof of Lemma 11.10 with individual morphisms replaced by $\underline{\mathrm{Hom}}$-complexes. We have maps

$$\underline{\mathrm{Hom}}(P_1, P_2) \xrightarrow[\mathrm{qis}]{\beta_*} \underline{\mathrm{Hom}}(P_1, X) \to \underline{\mathrm{Hom}}(P_1, \mathrm{hCoker}(\beta)) = \mathrm{hCoker}(\beta_*),$$

where β_* is given by composition with β, and the final identification is for formal reasons. Since P_2 is projective, the last complex is acyclic (by definition), so ψ is a quasi-isomorphism by Corollary 10.11, hence an isomorphism on homotopy classes of maps. In particular,

$$H^0\underline{\mathrm{Hom}}(P_1, P_2) = H^0\underline{\mathrm{Hom}}(P_2, X),$$

so since we have a given map in $H^0\underline{\mathrm{Hom}}(P_2, X)$, the induced map in $H^0\underline{\mathrm{Hom}}(P_1, P_2)$ is well-defined up to homotopy (as noted in the discussion following Definition 11.1). $\square$

In fact, we can show that all such homotopies between homotopies are homotopic, and so on, so this is the best outcome we could possibly hope for in establishing uniqueness.

Definition 11.12. The *ith* Ext-*group* of two chain complexes of A-modules M and N is defined as $\mathrm{Ext}^i_A(M,N) := H^i\underline{\mathrm{Hom}}(P,N)$, where P is some projective resolution of M.

As we just showed, this definition is independent of which P we choose.