# 25 The ring of adeles, strong approximation

## 25.1 Introduction to adelic rings

Recall that we have a canonical injection

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p,$$

that embeds $\mathbb{Z}$ into the product of its nonarchimedean completions. Each of the rings $\mathbb{Z}_p$ is compact, hence $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is compact (by Tychonoff's theorem). If we consider the analogous product $\prod_p \mathbb{Q}_p$ of the completions of $\mathbb{Q}$, each of the local fields $\mathbb{Q}_p$ is locally compact (as is $\mathbb{Q}_\infty = \mathbb{R}$), but the product $\prod_p \mathbb{Q}_p$ is **not locally compact**.

To see where the problem arises, recall that for any family of topological spaces $(X_i)_{i\in I}$ (where the index set $I$ is any set), the product topology on $X := \prod X_i$ is defined as the weakest topology that makes all the projection maps $\pi_i\colon X \to X_i$ continuous; it is thus generated by open sets of the form $\pi_i^{-1}(U_i)$ with $U_i \subseteq X_i$ open. Every open set in $X$ is a (possibly empty or infinite) union of open sets of the form

$$\prod_{i\in S} U_i \times \prod_{i\notin S} X_i,$$

with $S \subseteq I$ finite and each $U_i \subseteq X_i$ open (these sets form a *basis* for the topology on $X$). In particular, every open $U \subseteq X$ satisfies $\pi_i(U) = X_i$ for all but finitely many $i \in I$. Unless all but finitely many of the $X_i$ are compact, the space $X$ cannot possibly be locally compact for the simple reason that no compact set $C$ in $X$ contains a nonempty open set (if it did then we would have $\pi_i(C) = X_i$ compact for all but finitely many $i \in I$). Recall that to be locally compact means that for every $x \in X$ there is an open $U$ and compact $C$ such that $x \in U \subseteq C$.

To address this issue we want to take the product of the fields $\mathbb{Q}_p$ (or more generally, the completions of any global field) in a different way, one that yields a locally compact topological ring. This is the motivation of the *restricted product*, a topological construction that was invented primarily for the purpose of solving this number-theoretic problem.

## 25.2 Restricted products

This section is purely about the topology of restricted products; readers already familiar with restricted products should feel free to skip to the next section.

**Definition 25.1.** Let $(X_i)$ be a family of topological spaces indexed by $i \in I$, and let $(U_i)$ be a family of open sets $U_i \subseteq X_i$. The *restricted product* $\prod(X_i, U_i)$ is the topological space

$$\prod(X_i, U_i) := \{(x_i) : x_i \in U_i \text{ for almost all } i \in I\} \subseteq \prod X_i$$

with the basis of open sets

$$\mathcal{B} := \left\{\prod V_i : V_i \subseteq X_i \text{ is open for all } i \in I \text{ and } V_i = U_i \text{ for almost all } i \in I\right\},$$

where *almost all* means all but finitely many.

For each $i \in I$ we have a projection map $\pi_i\colon \prod(X_i, U_i) \to X_i$ defined by $(x_i) \mapsto x_i$; each $\pi_i$ is continuous, since if $W_i$ is an open subset of $X_i$, then $\pi_i^{-1}(W_i)$ is the union of all basic open sets $\prod V_i \in \mathcal{B}$ with $V_i = W_i$, which is an open set.

As sets, we always have

$$\prod U_i \subseteq \prod(X_i, U_i) \subseteq \prod X_i,$$

but in general the restricted product topology on $\prod(X_i, U_i)$ is not the same as the subspace topology it inherits from $\prod X_i$; it has more open sets. For example, $\prod U_i$ is an open set in $\prod(X_i, U_i)$, but unless $U_i = X_i$ for almost all $i$ (in which case $\prod(X_i, U_i) = \prod X_i$), it is not open in $\prod X_i$, and it is not open in the subspace topology on $\prod(X_i, U_i)$ because it does not contain the intersection of $\prod(X_i, U_i)$ with any basic open set in $\prod X_i$.

Thus the restricted product is a strict generalization of the direct product; the two coincide if and only if $U_i = X_i$ for almost all $i$. This is automatically true whenever the index set $I$ is finite, so only infinite restricted products are of independent interest.

**Remark 25.2.** The restricted product does not depend on any particular $U_i$. Indeed,

$$\prod(X_i, U_i) = \prod(X_i, U_i')$$

whenever $U_i' = U_i$ for almost all $i$; note that the two restricted products are not merely isomorphic, they are identical, both as sets and as topological spaces. It is thus enough to specify the $U_i$ for all but finitely many $i \in I$.

Each $x \in X := \prod(X_i, U_i)$ determines a (possibly empty) finite set

$$S(x) := \{i \in I : x_i \notin U_i\}.$$

Given any finite $S \subseteq I$, let us define

$$X_S := \{x \in X : S(x) \subseteq S\} = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i.$$

Notice that $X_S \in \mathcal{B}$ is an open set, and we can view it as a topological space in two ways, both as a subspace of $X$ or as a direct product of certain $X_i$ and $U_i$. Restricting the basis $\mathcal{B}$ for $X$ to a basis for the subspace $X_S$ yields

$$\mathcal{B}_S := \left\{\prod V_i : V_i \subseteq \pi_i(X_S) \text{ is open and } V_i = U_i = \pi_i(X_S) \text{ for almost all } i \in I\right\},$$

which is the standard basis for the product topology, so the two topologies on $X_S$ coincide.

We have $X_S \subseteq X_T$ whenever $S \subseteq T$, thus if we partially order the finite subsets $S \subseteq I$ by inclusion, the family of topological spaces $\{X_S : S \subseteq I \text{ finite}\}$ with inclusion maps $\{i_{ST}\colon X_S \hookrightarrow X_T | S \subseteq T\}$ forms a *direct system*, and we have a corresponding *direct limit*

$$\varinjlim_S X_S := \coprod X_S / \sim,$$

which is the quotient of the coproduct space (disjoint union) $\coprod X_S$ by the equivalence relation $x \sim i_{ST}(x)$ for all $x \in S \subseteq T$.[1] This direct limit is canonically isomorphic to the restricted product $X$, which gives us another way to define the restricted product; before proving this let us recall the general definition of a direct limit of topological spaces.

[1] The topology on $\coprod X_S$ is the weakest topology that makes the injections $X_S \hookrightarrow \coprod X_S$ continuous; its open sets are disjoint unions of open sets in the $X_S$. The topology on $\coprod X_S/\sim$ is the weakest topology that makes the quotient map $\coprod X_S \to \coprod X_S/\sim$ continuous; its open sets are images of open sets in $\coprod X_S$.

**Definition 25.3.** A *direct system* (or *inductive system*) in a category is a family of objects $\{X_i : i \in I\}$ indexed by a directed set $I$ (see Definition 8.7) and a family of morphisms $\{f_{ij}\colon X_i \to X_j : i \le j\}$ such that each $f_{ii}$ is the identity and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \le j \le k$.

**Definition 25.4.** Let $(X_i, f_{ij})$ be a direct system of topological spaces. The *direct limit* (or *inductive limit*) of $(X_i, f_{ij})$ is the quotient space

$$X = \varinjlim X_i := \coprod_{i \in I} X_i / \sim,$$

where $x_i \sim f_{ij}(x_i)$ for all $i \le j$. It is equipped with continuous maps $\phi_i\colon X_i \to X$ that are compositions of the inclusion maps $X_i \hookrightarrow \coprod X_i$ and quotient maps $\coprod X_i \twoheadrightarrow \coprod X_i/\sim$ and satisfy $\phi_i = \phi_j \circ f_{ij}$ for $i \le j$.

The topological space $X = \varinjlim X_i$ has the universal property that if $Y$ is another topological space with continuous maps $\psi_i\colon X_i \to Y$ that satisfy $\psi_i = \psi_j \circ f_{ij}$ for $i \le j$, then there is a unique continuous map $X \to Y$ for which all of the diagrams

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & X_j \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& X \arrow[d, dashed, "\exists!"] & \\
& Y &
\end{tikzcd}$$

commute (this universal property defines the direct limit in any category with coproducts).

We now prove that that $\prod(X_i, U_i) \simeq \varinjlim X_S$ as claimed above.

**Proposition 25.5.** *Let $(X_i)$ be a family of topological spaces indexed by $i \in I$, let $(U_i)$ be a family of open sets $U_i \subseteq X_i$, and let $X := \prod(X_i, U_i)$ be the corresponding restricted product. For each finite $S \subseteq I$ define*

$$X_S := \prod_{i \in S} X_i \times \prod_{i \notin S} U_i \subseteq X,$$

*and inclusion maps $i_{ST}\colon X_S \hookrightarrow X_T$, and let $\varinjlim X_S$ be the corresponding direct limit.*

*There is a canonical homeomorphism of topological spaces*

$$\varphi\colon X \xrightarrow{\sim} \varinjlim X_S$$

*that sends $x \in X$ to the equivalence class of $x \in X_{S(x)} \subseteq \coprod X_S$ in $\varinjlim X_S := \coprod X_S/\sim$, where $S(x) := \{i \in I : x_i \notin U_i\}$.*

*Proof.* To prove that the map $\varphi\colon X \to \varinjlim X_S$ is a homeomorphism, we need to show that it is (1) a bijection, (2) continuous, and (3) an open map.

(1) For each equivalence class $\mathcal{C} \in \varinjlim X_S := \coprod X_S/\sim$, let $S(\mathcal{C})$ be the intersection of all the sets $S$ for which $\mathcal{C}$ contains an element of $\coprod X_S$ in $X_S$. Then $S(x) = S(\mathcal{C})$ for all $x \in \mathcal{C}$, and $\mathcal{C}$ contains a unique element for which $x \in X_{S(x)} \subseteq \coprod X_S$ (distinct $x, y \in X_S$ cannot be equivalent). Thus $\varphi$ is a bijection.

(2) Let $U$ be an open set in $\varinjlim X_S = \coprod X_S/\sim$. The inverse image $V$ of $U$ in $\coprod X_S$ is open, as are the inverse images $V_S$ of $V$ under the canonical injections $\iota\colon X_S \hookrightarrow \coprod X_S$. The union of the $V_S$ in $X$ is equal to $\varphi^{-1}(U)$ and is an open set in $X$; thus $\varphi$ is continuous.

(3) Let $U$ be an open set in $X$. Since the $X_S$ form an open cover of $X$, we can cover $U$ with open sets $U_S := U \cap X_S$, and then $\coprod U_S$ is an open set in $\coprod X_S$. Moreover, for each $x \in \coprod U_S$, if $y \sim x$ for some $y \in \coprod X_S$ then $y$ and $x$ must correspond to the same element in $U$; in particular, $y \in \coprod U_S$, so $\coprod U_S$ is a union of equivalence classes in $\coprod X_S$. It follows that its image in $\varinjlim X_S = \coprod X_S/\sim$ is open. $\square$

Proposition 25.5 gives us another way to construct the restricted product $\prod(X_i, U_i)$: rather than defining it as a subset of $\prod X_i$ with a modified topology, we can instead construct it as a limit of direct products that are subspaces of $\prod X_i$.

We now specialize to the case of interest, where we are forming a restricted product using a family $(X_i)_{i\in I}$ of locally compact spaces and a family of open subsets $(U_i)$ that are almost all compact. Under these conditions the restricted product $\prod(X_i, U_i)$ is locally compact, even though the product $\prod X_i$ is not unless the index set $I$ is finite.

**Proposition 25.6.** *Let $(X_i)_{i\in I}$ be a family of locally compact topological spaces and let $(U_i)_{i\in I}$ be a corresponding family of open subsets $U_i \subseteq X_i$ almost all of which are compact. Then the restricted product $X := \prod(X_i, U_i)$ is locally compact.*

*Proof.* We first note that for each finite set $S \subseteq I$ the topological space

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} U_i$$

can be viewed as a finite product of locally compact spaces, since all but finitely many $U_i$ are compact, and the product of these is compact (by Tychonoff's theorem), hence locally compact. A finite product of locally compact spaces is locally compact, since we can construct compact neighborhoods as products of compact neighborhoods in each factor (in a finite product, products of open sets are open and products of compact sets are compact); thus the $X_S$ are locally compact, and they cover $X$ (since each $x \in X$ lies in $X_{S(x)}$). It follows that $X$ is locally compact, since each $x \in X_S$ has a compact neighborhood $x \in U \subseteq C \subseteq X_S$ that is also a compact neighborhood in $X$ (the image of $C$ under the inclusion map $X_S \to X$ is certainly compact, and $U$ is open in $X$ because $X_S$ is open in $X$). $\square$

## 25.3 The ring of adeles

Recall that for a global field $K$ (a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$), we use $M_K$ to denote the set of places of $K$ (equivalence classes of absolute values), and for any $v \in M_K$ we use $K_v$ to denote the corresponding local field (the completion of $K$ with respect to $v$). When $v$ is nonarchimedean we use $\mathcal{O}_v$ to denote the valuation ring of $K_v$, and for archimedean $v$ we define $\mathcal{O}_v := K_v$.[2]

**Definition 25.7.** Let $K$ be a global field. The *adele ring*[3] of $K$ is the restricted product

$$\mathbb{A}_K := \prod(K_v, \mathcal{O}_v)_{v\in M_K},$$

which we may view as a subset (but not a subspace!) of $\prod_v K_v$; indeed

$$\mathbb{A}_K = \left\{(a_v) \in \prod K_v : a_v \in \mathcal{O}_v \text{ for almost all } v\right\}.$$

[2]Per Remark 25.2, as far as the topology goes it doesn't matter how we define $\mathcal{O}_v$ at the finite number of archimedean places, but we would like each $\mathcal{O}_v$ to be a topological ring, which motivates this choice.

[3]In French one writes *adèle*, but it is common practice to omit the accent when writing in English.

For each $a \in \mathbb{A}_K$ we use $a_v$ to denote its projection in $K_v$; we make $\mathbb{A}_K$ a ring by defining addition and multiplication component-wise.

For each finite set of places $S$ we have the subring of *$S$-adeles*

$$\mathbb{A}_{K,S} := \prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v,$$

which is a direct product of topological rings. By Proposition 25.5, $\mathbb{A}_K \simeq \varinjlim \mathbb{A}_{K,S}$ is the direct limit of the $S$-adele rings, which makes it clear that $\mathbb{A}_K$ is also a topological ring.[4]

The canonical embeddings $K \hookrightarrow K_v$ induce a canonical embedding

$$\begin{aligned} K &\hookrightarrow \mathbb{A}_K \\ x &\mapsto (x, x, x, \ldots). \end{aligned}$$

Note that for each $x \in K$ we have $x \in \mathcal{O}_v$ for all but finitely many $v$. The image of $K$ in $\mathbb{A}_K$ is the subring of *principal adeles* (which of course is also a field).

We extend the normalized absolute value $\| \ \|_v$ of $K_v$ (see Definition 13.17) to $\mathbb{A}_K$ via

$$\|a\|_v := \|a_v\|_v,$$

and define the *adelic absolute value* (or *adelic norm*)

$$\|a\| := \prod_{v\in M_K} \|a\|_v \in \mathbb{R}_{\geq 0}$$

which we note converges to zero unless $\|a\|_v = 1$ for all but finitely many $v$, in which case it is effectively a finite product.[5] For $\|a\| \neq 0$ this is equal to the size of the $M_K$-*divisor* $(\|a\|_v)$ we defined in Lecture 15 (see Definition 15.1). For any nonzero principal adele $a$, we have $a \in K^\times$ and $\|a\| = 1$, by the product formula (Theorem 13.21).

**Example 25.8.** For $K = \mathbb{Q}$ the adele ring $\mathbb{A}_\mathbb{Q}$ is the union of the rings

$$\mathbb{R} \times \prod_{p\in S} \mathbb{Q}_p \times \prod_{p\notin S} \mathbb{Z}_p$$

where $S$ varies over finite sets of primes (but note that the topology is the restricted product topology, not the subspace topology in $\prod_{p\leq\infty} \mathbb{Q}_p$). We can also write $\mathbb{A}_\mathbb{Q}$ as

$$\mathbb{A}_\mathbb{Q} = \left\{ a \in \prod_{p\leq\infty} \mathbb{Q}_p : \|a\|_p \leq 1 \text{ for almost all } p \right\}.$$

**Proposition 25.9.** *The adele ring $\mathbb{A}_K$ of a global field $K$ is locally compact and Hausdorff.*

*Proof.* Local compactness follows from Proposition 25.6, since the local fields $K_v$ are all locally compact and all but finitely many $\mathcal{O}_v$ are valuation rings of a nonarchimedean local field, hence compact ($\mathcal{O}_v = \{x \in K_v : \|x\|_v \leq 1\}$ is a closed ball). The product space $\prod_v K_v$ is Hausdorff, since each $K_v$ is Hausdorff, and the topology on $\mathbb{A}_K \subseteq \prod K_v$ is finer than the subspace topology, so $\mathbb{A}_K$ is also Hausdorff. $\square$

[4] By definition it is a topological space that is also a ring; to be a topological ring is a stronger condition (the ring operations must be continuous), but this property is preserved by direct limits so all is well.

[5] For $v \nmid \infty$, if $\|a\|_v < 1$ then $\|a\|_v \leq 1/2$, since $\|a\|_v := q^{-v(a_v)}$ for some prime power $q$.

Proposition 25.9 implies that the additive group of $\mathbb{A}_K$ (which is sometimes denoted $\mathbb{A}_K^+$ to emphasize that we are viewing it as a group rather than a ring) is a locally compact group, and therefore has a Haar measure that is unique up to scaling, by Theorem 13.14. Each of the completions $K_v$ is a local field with a Haar measure $\mu_v$, which we normalize as follows:

- $\mu_v(\mathcal{O}_v) = 1$ for all nonarchimedean $v$;
- $\mu_v(S) = \mu_{\mathbb{R}}(S)$ for $K_v \simeq \mathbb{R}$, where $\mu_{\mathbb{R}}(S)$ is the Lebesgue measure on $\mathbb{R}$;
- $\mu_v(S) = 2\mu_{\mathbb{C}}(S)$ for $K_v \simeq \mathbb{C}$, where $\mu_{\mathbb{C}}(S)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$.

Note that the normalization of $\mu_v$ at the archimedean places is consistent with the measure $\mu$ on $K_{\mathbb{R}} \simeq \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ induced by the canonical inner product on $K_{\mathbb{R}} \subseteq K_{\mathbb{C}}$ that we defined in Lecture 14 (see §14.2).

We now define a measure $\mu$ on $\mathbb{A}_K$ as follows. We take as a basis for the $\sigma$-algebra of measurable sets all sets of the form $\prod_v B_v$, where each $B_v$ is a measurable set in $K_v$ with $\mu_v(B_v) < \infty$ such that $B_v = \mathcal{O}_v$ for almost all $v$ (the $\sigma$-algebra is then generated by taking countable intersections, unions, and complements in $\mathbb{A}_K$). We then define

$$\mu\left(\prod_v B_v\right) := \prod_v \mu_v(B_v).$$

It is easy to verify that $\mu$ is a Radon measure, and it is clearly translation invariant since each of the Haar measures $\mu_v$ is translation invariant and addition is defined component-wise; note that for any $x \in \mathbb{A}_K$ and measurable set $B = \prod_v B_v$ the set $x + B = \prod_v (x_v + B_v)$ is also measurable, since $x_v + B_v = \mathcal{O}_v$ whenever $x_v \in \mathcal{O}_v$ and $B_v = \mathcal{O}_v$, and this applies to almost all $v$. It follows from uniqueness of the Haar measure (up to scaling) that $\mu$ is a Haar measure on $\mathbb{A}_K$ which we henceforth adopt as our normalized Haar measure on $\mathbb{A}_K$.

We now want to understand the behavior of the adele ring $\mathbb{A}_K$ under base change. Note that the canonical embedding $K \hookrightarrow \mathbb{A}_K$ makes $\mathbb{A}_K$ a $K$-vector space, and if $L/K$ is any finite separable extension of $K$ (also a $K$-vector space), we may consider the tensor product

$$\mathbb{A}_K \otimes_K L,$$

which is also an $L$-vector space. As a topological $K$-vector space, the topology on $\mathbb{A}_K \otimes L$ is just the product topology on $[L:K]$ copies of of $\mathbb{A}_K$ (this applies whenever we take a tensor product of topological vector spaces, one of which has finite dimension).

**Proposition 25.10.** *Let $L$ be a finite separable extension of a global field $K$. There is a natural isomorphism of topological rings*

$$\mathbb{A}_L \simeq \mathbb{A}_K \otimes_K L$$

*that makes the following diagram commute*

$$\begin{tikzcd} L \arrow[r, "\sim"] \arrow[d] & K \otimes_K L \arrow[d] \\ \mathbb{A}_L \arrow[r, "\sim"] & \mathbb{A}_K \otimes_K L \end{tikzcd}$$

*Proof.* On the RHS the tensor product $\mathbb{A}_K \otimes_K L$ is isomorphic to the restricted product

$$\prod_{v \in M_K} (K_v \otimes_K L,\, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L).$$

Explicitly, each element of $\mathbb{A}_K \otimes_K L$ is a finite sum of elements of the form $(a_v) \otimes x$, where $(a_v) \in \mathbb{A}_K$ and $x \in L$, and there is a natural isomorphism of topological rings

$$\begin{aligned} \mathbb{A}_K \otimes_K L &\xrightarrow{\sim} \prod_{v \in M_K} (K_v \otimes_K L,\, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L) \\ (a_v) \otimes x &\mapsto (a_v \otimes x). \end{aligned}$$

Here we are using the general fact that tensor products commute with direct limits (restricted direct products can be viewed as direct limits via Proposition 25.5).[6]

On the LHS we have $\mathbb{A}_L := \prod_{w \in M_L} (L_w, \mathcal{O}_w)$. But note that $K_v \otimes_K L \simeq \prod_{w|v} L_w$, by Theorem 11.23 and $\mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L \simeq \prod_{w|v} \mathcal{O}_w$, by Corollary 11.26. These isomorphisms preserve both the algebraic and the topological structures of both sides, and it follows that

$$\mathbb{A}_K \otimes_K L \simeq \prod_{v \in M_K} (K_v \otimes_K L,\, \mathcal{O}_v \otimes_{\mathcal{O}_K} \mathcal{O}_L) \simeq \prod_{w \in M_L} (L_w, \mathcal{O}_w) = \mathbb{A}_L$$

is an isomorphism of topological rings. The image of $x \in L$ in $\mathbb{A}_K \otimes_K L$ via the canonical embedding of $L$ into $\mathbb{A}_K \otimes_K L$ is $1 \otimes x = (1, 1, 1, \ldots) \otimes x$, whose image $(x, x, x, \ldots) \in \mathbb{A}_L$ is equal to the image of $x \in L$ under the canonical embedding of $L$ into its adele ring $\mathbb{A}_L$. $\square$

**Corollary 25.11.** *Let $L$ be a finite separable extension of a global field $K$ of degree $n$. There is a natural isomorphism of topological $K$-vector spaces (and locally compact groups)*

$$\mathbb{A}_L \simeq \mathbb{A}_K \oplus \cdots \oplus \mathbb{A}_K$$

*that identifies $\mathbb{A}_K$ with the direct sum of $n$ copies of $\mathbb{A}_K$, and this isomorphism restricts to an isomorphism $L \simeq K \oplus \cdots \oplus K$ of the principal adeles of $\mathbb{A}_L$ with the $n$-fold direct sum of the principal adeles of $\mathbb{A}_K$.*

**Theorem 25.12.** *For each global field $L$ the principal adeles $L \subseteq \mathbb{A}_L$ form a discrete cocompact subgroup of the additive group of the adele ring $\mathbb{A}_L$.*

*Proof.* Let $K$ be the rational subfield of $L$ (so $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$). It follows from Corollary 25.11 that if the theorem holds for $K$ then it holds for $L$, so we will prove the theorem for $K$. Let us identify $K$ with its image in $\mathbb{A}_K$ (the principal adeles).

To show that the topological group $K$ is discrete in the locally compact group $\mathbb{A}_K$, it suffices to show that 0 is an isolated point. Consider the open set

$$U = \{a \in \mathbb{A}_K : \|a\|_\infty < 1 \text{ and } \|a\|_v \le 1 \text{ for all } v < \infty\},$$

where $\infty$ denotes the unique infinite place of $K$ (either the real place of $\mathbb{Q}$ or the place corresponding to the nonarchimedean valuation $v_\infty(f/g) = \deg g - \deg f$ of $\mathbb{F}_q(t)$). The product formula (Theorem 13.21) implies $\|a\| = 1$ for all $a \in K^\times \subseteq \mathbb{A}_K$, so $U \cap K = \{0\}$.

To prove that the quotient $\mathbb{A}_K/K$ is compact, we consider the set

$$W := \{a \in \mathbb{A}_K : \|a\|_v \le 1 \text{ for all } v\}.$$

[6] In general, tensor products *do not* commute with infinite direct products; there is always a natural map $(\prod_n A_n) \otimes B \to \prod_n (A_n \otimes B)$, but it need be neither a monomorphism or an epimorphism. This is another motivation for using restricted direct products to define the adeles, so that base change works as it should.

If we let $U_\infty \coloneqq \{x \in K_\infty : \|x\|_\infty \le 1\}$, then

$$W = U_\infty \times \prod_{v<\infty} \mathcal{O}_v \subseteq \mathbb{A}_{K,\{\infty\}} \subseteq \mathbb{A}_K$$

is a product of compact sets and therefore compact. We will show that $W$ contains a complete set of coset representatives for $K$ in $\mathbb{A}_K$. This implies that $\mathbb{A}_K/K$ is the image of the compact set $W$ under the (continuous) quotient map $\mathbb{A}_K \to \mathbb{A}_K/K$, hence compact.

Let $a = (a_v)$ be any element of $\mathbb{A}_K$. We wish to show that $a = b + c$ for some $b \in W$ and $c \in K$, which we will do by constructing $c \in K$ so that $b = a - c \in W$.

For each $v < \infty$ define $x_v \in K$ as follows: put $x_v \coloneqq 0$ if $\|a_v\|_v \le 1$ (almost all $v$), and otherwise choose $x_v \in K$ so that $\|a_v - x_v\|_v \le 1$ and $\|x_v\|_w \le 1$ for $w \ne v$. To show that such an $x_v$ exists, let us first suppose $a_v = r/s \in K$ with $r, s \in \mathcal{O}_K$ coprime (note that $\mathcal{O}_K$ is a PID), and let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_v$. The ideals $\mathfrak{p}^{v(s)}$ and $\mathfrak{p}^{-v(s)}(s)$ are coprime, so we can write $r = r_1 + r_2$ with $r_1 \in \mathfrak{p}^{v(s)}$ and $r_2 \in \mathfrak{p}^{-v(s)}(s) \subseteq \mathcal{O}_K$, so that $a_v = r_1/s + r_2/s$ with $v(r_1/s) \ge 0$ and $w(r_2/s) \ge 0$ for all $w \ne v$. If we now put $x_v \coloneqq r_2/s$, then $\|a_v - x_v\|_v = \|r_1/s\|_v \le 1$ and $\|x_v\|_w = \|r_2/s\|_w \le 1$ for all $w \ne v$ as desired. We can approximate any $a_v' \in K_v$ by such an $a_v \in K$ with $\|a_v' - a_v\|_v < \epsilon$ and construct $x_v$ as above so that $\|a_v - x_v\|_v \le 1$ and $\|a_v' - x_v\|_v \le 1 + \epsilon$; but for sufficiently small $\epsilon$ this implies $\|a_v' - x_v\|_v \le 1$, since the nonarchimedean absolute value $\| \; \|_v$ is discrete.

Finally, let $x \coloneqq \sum_{v<\infty} x_v \in K$ and choose $x_\infty \in \mathcal{O}_K$ so that

$$\|a_\infty - x - x_\infty\|_\infty \le 1.$$

For $a_\infty - x \in \mathbb{Q}_\infty \simeq \mathbb{R}$, we can take $x_\infty \in \mathbb{Z}$ in the real interval $[a_\infty - x - 1, a_\infty - x + 1)$. For $a_\infty - x \in \mathbb{F}_q(t)_\infty \simeq \mathbb{F}_q((t^{-1}))$ we can take $x_\infty \in \mathbb{F}_q[t]$ to be the polynomial of least degree for which $a_\infty - x - x_\infty \in \mathbb{F}_q[[t^{-1}]]$.[7]

Now let $c \coloneqq \sum_{v\le\infty} x_v \in K \subseteq \mathbb{A}_K$, and let $b \coloneqq a - c$. Then $a = b + c$, with $c \in K$, and we claim that $b \in W$. For each $v < \infty$ we have $x_w \in \mathcal{O}_v$ for all $w \ne v$ and

$$\|b\|_v = \|a - c\|_v = \left\| a_v - \sum_{w\le\infty} x_w \right\|_v \le \max\left(\|a_v - x_v\|_v, \max(\{\|x_w\|_v : w \ne v\})\right) \le 1,$$

by the nonarchimedean triangle inequality. For $v = \infty$ we have $\|b\|_\infty = \|a_\infty - c\|_\infty \le 1$ by our choice of $x_\infty$, and $\|b\|_v \le 1$ for all $v$, so $b \in W$ as claimed and the theorem follows. $\square$

**Corollary 25.13.** *For any global field $K$ the quotient $\mathbb{A}_K/K$ is a compact group.*

*Proof.* As explained in Remark 14.4, this follows immediately (in particular, the fact that $K$ is a discrete subgroup of $\mathbb{A}_K$ implies that it is closed and therefore $\mathbb{A}_K/K$ is Hausdorff). $\square$

## 25.4 Strong approximation

We are now ready to prove the strong approximation theorem, an important result that has many applications. We begin with an adelic version of the Blichfeldt-Minkowski lemma.

[7]Note that while $\mathbb{F}_q((t^{-1})) \simeq \mathbb{F}_q((t))$, in order to view $K = \mathbb{F}_q(t)$ as canonically embedded in its completion with respect to the absolute value $|f|_\infty = q^{\deg f}$ we need to view $K_\infty$ as the field of Laurent series in a uniformizer, which we may take to be $t^{-1}$ (but not $t$), and the valuation ring of $K_\infty$ is $\mathbb{F}_q[[-t]]$ (not $\mathbb{F}_q[[t]]$).

**Lemma 25.14** (Adelic Blichfeldt-Minkowski lemma)**.** *Let $K$ be a global field. There is a positive constant $B_K$ such that for any $a \in \mathbb{A}_K$ with $\|a\| > B_K$ there exists a nonzero principal adele $x \in K \subseteq \mathbb{A}_K$ for which $\|x\|_v \le \|a\|_v$ for all $v \in M_K$.*

*Proof.* Let $b_0 := \operatorname{covol}(K)$ be the measure of a fundamental region for $K$ in $\mathbb{A}_K$ under our normalized Haar measure $\mu$ on $\mathbb{A}_K$ (by Theorem 25.12, $K$ is cocompact, so $b_0$ is finite). Now define

$$b_1 := \mu\left(\left\{z \in \mathbb{A}_K : \|z\|_v \le 1 \text{ for all } v \text{ and } \|z\|_v \le \tfrac{1}{4} \text{ if } v \text{ is archimedean}\right\}\right).$$

Then $b_1 \neq 0$, since $K$ has only finitely many archimedean places. Now let $B_K := b_0/b_1$.

Suppose $a \in \mathbb{A}_K$ satisfies $\|a\| > B_K$. We know that $\|a\|_v \le 1$ for all almost all $v$, so $\|a\| \neq 0$ implies that $\|a\|_v = 1$ for almost all $v$. Let us now consider the set

$$T := \left\{t \in \mathbb{A}_K : \|t\|_v \le \|a\|_v \text{ for all } v \text{ and } \|t\|_v \le \tfrac{1}{4}\|a\|_v \text{ if } v \text{ is archimedean}\right\}.$$

From the definition of $b_1$ we have

$$\mu(T) = b_1\|a\| > b_1 B_K = b_0;$$

this follows from the fact that the Haar measure on $\mathbb{A}_K$ is the product of the normalized Haar measures $\mu_v$ on each of the $K_v$. Since $\mu(T) > b_0$, the set $T$ is not contained in any fundamental region for $K$, so there must be distinct $t_1, t_2 \in T$ with the same image in $\mathbb{A}_K/K$, equivalently, whose difference $x = t_1 - t_2$ is a nonzero element of $K \subseteq \mathbb{A}_K$. We have

$$\|t_1 - t_2\|_v \le \begin{cases} \max(\|t_1\|_v, \|t_2\|_v) \le \|a\|_v & \text{nonarch. } v;\\ \|t_1\|_v + \|t_2\|_v \le 2\cdot\frac{1}{4}\|a\|_v \le \|a\|_v & \text{real } v;\\ (\|t_1 - t_2\|_v^{1/2})^2 \le (\|t_1\|_v^{1/2} + \|t_2\|_v^{1/2})^2 \le (2\cdot\frac{1}{2}\|a\|_v^{1/2})^2 \le \|a\|_v & \text{complex } v.\end{cases}$$

Here we have used the fact that the normalized absolute value $\|\ \|_v$ satisfies the nonarchimedean triangle inequality when $v$ is nonarchimedean, $\|\ \|_v$ satisfies the archimedean triangle inequality when $v$ is real, and $\|\ \|_v^{1/2}$ satisfies the archimedean triangle inequality when $v$ is complex. Thus $\|x\|_v = \|t_1 - t_2\|_v \le \|a\|_v$ for all places $v \in M_K$ as desired. $\square$

**Remark 25.15.** Lemma 25.14 should be viewed as an analog of Mikowski's lattice point theorem (Thoerem 14.12) and a generalization of Proposition 15.9. In Theorem 14.12 we have a discrete cocompact subgroup $\Lambda$ in a real vector space $V \simeq \mathbb{R}^n$ and a sufficiently large symmetric convex set $S$ that must contain a nonzero element of $\Lambda$. In Lemma 25.14 the lattice $\Lambda$ is replaced by $K$, the vector space $V$ is replaced by $\mathbb{A}_K$, the symmetric convex set $S$ is replaced by the set

$$L(a) \coloneqq \{x \in \mathbb{A}_K : \|x\|_v \le \|a\|_v \text{ for all } v \in M_K\},$$

and sufficiently large means $\|a\| > B_K$, putting a lower bound on $\mu(L(a))$. Proposition 15.9 is actually equivalent to Lemma 25.14 in the case that $K$ is a number field: use the $M_K$-divisor $c \coloneqq (\|a\|_v)$ and note that $L(c) = L(a) \cap K$.

**Theorem 25.16** (Strong Approximation)**.** *Let $M_K = S \sqcup T \sqcup \{w\}$ be a partition of the places of a global field $K$ with $S$ finite. Fix $a_v \in K$ and $\epsilon_v \in \mathbb{R}_{>0}$ for each $v \in S$. There exists an $x \in K$ for which*

$$\begin{aligned} \|x - a_v\|_v &\le \epsilon_v \text{ for all } v \in S,\\ \|x\|_v &\le 1 \text{ for all } v \in T, \end{aligned}$$

*(note that there is no constraint on $\|x\|_w$).*

*Proof.* Let $W = \{z \in \mathbb{A}_K : \|z\|_v \le 1 \text{ for all } v \in M_K\}$ as in the proof of Theorem 25.12. Then $W$ contains a complete set of coset representatives for $K \subseteq \mathbb{A}_K$, so $\mathbb{A}_K = K + W$. For any nonzero $u \in K \subseteq \mathbb{A}_K$ we also have $\mathbb{A}_K = K + uW$: given $c \in \mathbb{A}_K$ write $u^{-1}c \in \mathbb{A}_K$ as $u^{-1}c = a + b$ with $a \in K$ and $b \in W$ and then $c = ua + ub$ with $ua \in K$ and $ub \in uW$. Now choose $z \in \mathbb{A}_K$ such that

$$0 < \|z\|_v \le \epsilon_v \text{ for } v \in S, \quad 0 < \|z\|_v \le 1 \text{ for } v \in T, \quad \|z\|_w > B \prod_{v \ne w} \|z\|_v^{-1},$$

where $B$ is the constant in the Blichfeldt-Minkowski Lemma 25.14 (this is clearly possible: every $z = (z_v)$ with $\|z_v\|_v \le 1$ is an element of $\mathbb{A}_K$). We have $\|z\| > B$, so there is a nonzero $u \in K \subseteq \mathbb{A}_K$ with $\|u\|_v \le \|z\|_v$ for all $v \in M_K$.

Now let $a = (a_v) \in \mathbb{A}_K$ be the adele with $a_v$ given by the hypothesis of the theorem for $v \in S$ and $a_v = 0$ for $v \notin S$. We have $\mathbb{A}_K = K + uW$, so $a = x + y$ for some $x \in K$ and $y \in uW$. Therefore

$$\|x - a_v\|_v = \|y\|_v \le \|u\|_v \le \|z\|_v \le \begin{cases} \epsilon_v & \text{for } v \in S, \\ 1 & \text{for } v \in T, \end{cases}$$

as desired. $\square$

**Corollary 25.17.** *Let $K$ be a global field and let $w$ be any place of $K$. Then $K$ is dense in the restricted product $\prod_{v \ne w}(K_v, \mathcal{O}_v)$.*

**Remark 25.18.** Theorem 25.16 and Corollary 25.17 can be generalized to algebraic groups; see [1] for a survey.

## References


[1] Andrei S. Rapinchuk, *Strong approximation for algebraic groups*, Thin groups and superstrong approximation, MSRI Publications **61**, 2013.