Sophus Lie, the mathematician

by
Sigurdur Helgason

Sophus Lie is known among all mathematicians as the founder of the theory of transformation groups which then gave rise to the modern theory of the so-called Lie groups. This is in fact a subject which permeates many branches of modern mathematics and mathematical physics. However in order to see Lie's work in perspective it is important to realize that his first mathematical love was geometry and that this attachment remained with him all his life. I shall therefore try to describe at least some of his work in chronological order; in this way we can better understand the motivations to various stages in his work and see why his ideas took a long time to be assimilated by the mathematical public, at least longer time than he himself would have liked.

Since my remarks are aimed at a general audience I have kept mathematical technicalities at a minimum. Other lectures at this conference will deal with Lie's work in considerable detail so I shall confine myself to selected highlights.

In view of the intensity with which Lie carried on his mathematical investigations in his later career it is remarkable that his calling to research in mathematics did not take place until 1868 (age 26), three years after he had finished his university studies in the natural sciences. Yet once in 1862, Sylow was a substitute teacher at the university and gave a lecture series on Galois Theory. At that time this does not seem to have made a very deep impression on Lie, but later developments suggest that it was well etched in his memory. Sylow relates that in 1868, Lie asked him to lend him his old lecture notes on Galois Theory.

Considering the fact that his research period was limited to 30 years, his mathematical production is most impressive; yet many unwritten papers and unworked ideas went with him to the grave. Nevertheless, in contrast to say Gauss, he was generally eager to publish his ideas and results quickly. The presentation often suffered because of this (he himself would on occasion admit to "leichtsinniger Darstellung" about some of his papers) and this meant that some of his ideas would first be understood by the mathematical public through works of others. This was often a source of disappointment to him; thus in a letter to Mittag-Leffler 1882 he writes: "It is strange that those of my works, which are without comparison the most original and far-reaching, receive much less attention than the less important ones", not an unusual phenomenon.

During the three years after his university degree he gave private lessons in mathematics and once gave a series of popular lectures on astronomy. At the same time he speculated a good deal about geometry and in 1868 he came across the works of J. Poncelet and J. Plücker. These seem to have inspired Lie in a very decisive fashion. These works were:

J. Poncelet: (i) Traité des propriétés projectives des figures (1822)
(ii) Théorie des polaires réciproques (Crelle's Journal 1829).
J. Plücker: System der Geometrie des Raumes in neuer analytischer Behandlungsweise (Crelle's Journal 1846).

Poncelet's 1822 book originated to some extent in the prison at Saratov near Volga; Poncelet was an officer in Napoleon's Russia campaign and was captured. One of the innovations of Poncelet's was an introduction and use of complex numbers in projective geometry; thereby a nondegenerate conic in homogeneous coordinates

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$$

and a line $ax + by + cz = 0$ will always intersect in two points.

In Plücker's paper geometric figures are no longer a collection of points but geometry becomes just as much a study of families of lines or of spheres etc. To explain this we consider two points in space with homogeneous coordinates

$$X = (x_1, x_2, x_3, x_4), \qquad Y = (y_1, y_2, y_3, y_4)$$

and put $p_{ik} = x_i y_k - y_i x_k$. Up to a factor the six p_{ik} are the homogeneous coordinates of the line joining X and Y (independent of the choice of coordinates (x_i) and (y_j) and they satisfy the identity

$$P = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0\,.$$

Thus the 4-dimensional set of lines in the three-space is represented as a quadric of signature (3,3) in five-dimensional projective space. Two lines intersect if the corresponding points on the quadric lie on the same line in the quadric. One can now study the geometric meaning of equations like $\sum a_{ik}p_{ik} = 0$ etc. and this represents the rudiments of Plücker's line geometry. Curves, surfaces and three-dimensional bodies in point geometry are replaced by their respective analogs in line geometry, called *line surfaces, line congruences* and *line complexes.* For example

$$(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}) = (t, 0, 1, t, 0, -t^2)$$

is one of the families of generating lines for a hyperboloid. It is a line surface. The set of tangents to a surface is a line complex.

Lie retained a life long admiration for Poncelet and Plücker, whom he never met, they having died in 1867 and 1868, respectively. Nevertheless, Lie considered himself a student of Plücker's.

In 1869 Lie writes his first paper "Representation der Imaginären der Plangeometrie" Here he considers a mapping T given by

$$T(x + yi, z + pi) = (x, y, z),$$

the "weight" p being fixed. According to Lie, "Every theorem in plane geometry should thereby be special case of a stereometric double theorem in the geometry of line congruences". Lie published the paper privately and sent a copy along with an application to the Collegium Academicum in Christiania for a travelling fellowship. He was also eager to assure himself of priority for these discoveries. Although few seem to have understood the work (it was really a tersely written research announcement), Lie had already earned a certain reputation and the request for a fellowship was granted, enabling Lie to visit the mathematical capitals of Europe. In Lie's collected works edited by Engel and Heegaard this 8 page paper (together with its expanded form) is given a commentary of over 100 pages.

Having received the fellowship Lie spent the winter 1869–1870 in Berlin where he came into close contact with Felix Klein and they became good friends. Klein was very much involved in line geometry at the time, having been a student of Plücker's. Klein had a great talent for absorbing new ideas and finding interconnection between them; Lie was more immersed in the wealth of his own ideas and perhaps felt the need to catch up in his research. After all, Klein had already obtained his doctor's degree whereas Lie had not, although he was seven years older. In Kummer's seminar, Lie gave several lectures on his work on the Reye complex (the set of lines intersecting the faces of a tetrahedron in a given cross ratio), an outgrowth of his first paper. For example he proved that it is an orbit under a certain group fixing the vertices of the polyhedron. This gave rise to joint work of Klein and Lie on what they called W-curves (curves whose tangents are contained in the Reye complex). In the Spring of 1870 they both travelled to Paris; taking adjoining rooms in a hotel. In Paris they got acquainted with the group theorist Camille Jordan and the differential geometer Gaston Darboux. Jordan had just published his paper "Memoire sur les groupes de mouvements" which undoubtedly provided Klein and Lie with much stimulus for the transformation group viewpoint in their work.

It was in early July 1870 that Lie made his most famous geometric discovery, namely what is now called Lie's line-sphere transformation. I would like to comment on this in some detail.

For this Lie defines a "sphere geometry" à la Plücker's line geometry. Consider a sphere with the equation

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + D = 0.$$

In addition to a, b, c, D we introduce a fifth quantity, the radius r satisfying

$$r^2 = a^2 + b^2 + c^2 - D\,.$$

These quantities are to be considered as the coordinates in the space of spheres. We now introduce the corresponding homogeneous coordinates by

$$a = \xi/\nu, \quad b = \eta/\nu, \quad c = \zeta/\nu, \quad r = \lambda/\nu, \quad D = \mu/\nu\,,$$

and then we have the connection

$$\Phi = \xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \mu\nu = 0.$$

Note that points are included ($\lambda = 0$) and so are hyperplanes (for $\nu = 0$).

Recall now the Plücker coordinates (p_{ik}) in line geometry with the connection $P = 0$. Given the sphere coordinates above we define homogeneous line coordinates (p_{ik}) by

$$\begin{aligned} p_{12} &= \xi + i\eta, & p_{34} &= \xi - i\eta, \\ p_{13} &= \zeta + \lambda, & p_{42} &= \zeta - \lambda, \\ p_{14} &= \mu, & p_{23} &= -\nu. \end{aligned}$$

Then $P = 0$ as a consequence of the relation $\Phi = 0$.

This is Lie's line-sphere transformation in analytic formulation. It has the crucial property that intersecting lines correspond to spheres that are tangential.

In the transformation above the coordinates are necessarily complex. The group $\mathbf{SL}(4, \mathbf{C})$ acts via projective transformations on the space of lines in $\mathbf{C}^3$ and $\mathbf{SO}(6, \mathbf{C})$ acts on the projective bundle associated with the tangent bundle of $\mathbf{C}^3$. The Lie line-sphere transformation is a realization of the local isomorphism of these two groups.

Lie himself drew the following consequence of the transformation for surface theory where one has the concept of asymptotic lines (on which the second fundamental form vanishes) and the lines of curvature (curves whose directions are where the curvature is extremal) (see Lie [22], Vol. II, p. 35, and Darboux [11], Vol. I, p. 232):

To the set of tangents to a given surface S we associate as above the set of spheres tangent to another surface S'. To the subset of lines tangent to S at a point M corresponds the set of spheres tangent to S' at a point M'. If M then runs through an asymptotic line for S, M' runs through a line of curvature of S'.

Darboux, in the quoted classic work and Klein in 1893 (see [18]) refer to this as one of the most brilliant discoveries in differential geometry of those days. Lie himself was rather pleased and the place in his thesis where this application of the line-sphere transformation appears has the following motto:

> "Das Ziel der Wissenschaft ist einerseits, neue Tatsachen zu erobern, anderseits, bekannte unter höheren Gesichtpunkten zusammenfassen."

It should however be remembered that the mapping is based on complexifying some of the coordinates. In fact a surface may not have any asymptotic lines (they are in other words imaginary). Thus the surface theorem above does not figure prominently in modern differential geometry. In fact, Blaschke' s work [2], Vol. 3, is the last one that I have seen that treats it.

However, the transformation has a certain "real form" (see e.g. Cecil [10]): Consider the quadric

$$\mathbf{Q}^4 : -x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0$$

in real projective space $\mathbf{P}^5$. This quadric contains projective lines but no linear subspaces of higher dimension. The mapping of Lie is a bijection between the set of oriented spheres, oriented planes and point spheres in $\mathbf{R}^3 \cup \infty$ onto the set of points on $\mathbf{Q}^4$. Let $\cdot$ denote the dot product on $\mathbf{R}^3$.

$\mathbf{R}^3 \cup \{\infty\}$ $\quad$ $S_r(a)$: Sphere, center a, radius r

$\mathbf{Q}^4$ $\quad$ $P(a,r)$: Point $(1 + a \cdot a - r^2, 1 - a \cdot a + r^2, 2a_1, 2a_2, 2a_3, r)$.

The mapping $S_r(a) \to P(a,r)$ is a combination of a stereographic projection and an imbedding of $\mathbf{R}^4$ into $\mathbf{P}^4$ and of $\mathbf{R}^5$ into $\mathbf{P}^5$. Under this map a family of spheres with the same tangent plane at a point gets mapped to a line on $\mathbf{Q}^4$. Lie also proved that any line-preserving diffeomorphism of $\mathbf{Q}^4$ is the restriction to $\mathbf{Q}^4$ of a projective transformation of $\mathbf{P}^5$, i.e. a member of $\mathbf{PO}(4,2)$. Since the signature of the quadric $P = 0$ in line geometry was (3,3) and since the groups $\mathbf{PO}(3,3)$ and $\mathbf{PO}(4,2)$ are not locally isomorphic, real line geometry and real sphere geometry are different. However the groups have the same complexification.

As mentioned, Lie made this discovery in early July 1870. Klein, who lived in an adjoining room in the hotel recalls (cf. [19], Vol. 1, p. 97):

> "...one morning I got up early and wanted to go out right away when Lie, who still lay in bed called me into his room. He explained to me the relationship he had found during the night between the asymptotic curves of one surface and the lines of curvature of another, but in such a way that I could not understand a word. In any case, he assured me that

> this meant that the asymptotic curves of the fourth degree Kummer surface must be algebraic curves of degree sixteen. That morning, while I was visiting the *Conservatoire des Arts et Métiers*, the thought came to me that these must be the same curves of degree sixteen that had appeared in my paper "Theorien der Liniencomplexe ersten und zweiten grades" and I quickly succeeded in showing this independently of Lie's geometric considerations. When I returned around four o'clock in the afternoon, Lie had gone out, so I left him a summary of my results in a letter".

Later this was the subject of a joint paper of Klein and Lie. In the meantime Lie summarized his results in a letter to the Scientific Society in Christiania, starting:

> "I take the liberty of communicating the following scientific results with the purpose of possibly securing my priority".

Shortly after this (July 17) the Franco-German war broke out and Klein had to return to Germany, but Lie stayed behind. Many mathematicians, after they have proved a good theorem, like to go for a long walk and contemplate consequences of the theorem and its relationship to known results. Lie was no exception and decided to do this in the grand style: take a hike from Paris to Italy. However, he had only reached Fountainbleu when he was arrested as a spy (having a map and some mysterious papers from Klein with him did not help), and was imprisoned for 4 weeks before Darboux got him released. As he later related to his regular correspondent Adolf Mayer this gave him plenty of peace and quiet to contemplate various ramifications of his discovery. He managed to reach Italy before the German armies surrounded Paris.

During the Spring 1871, Lie was a stipendiat at the University of Christiania, and did some teaching at his old Gymnasium (Nissen's). In July he submits his doctoral thesis "Über eine Classe geometrischer Transformationen", later published in a greatly expanded form in Math. Ann. "Über Complexe, insbesondere Linien- und Kugel- Complexe mit anvendung auf die Theorie partialler Differential-Gleichungen". In the thesis he mentions several elementary problems of the following type: How many spheres intersect five given spheres under the same angle? Using his line-sphere transformation he transfers this to a problem in line geometry and comes up with the number 16 but here he is talking about spheres whose coordinates are complex.

Since the line-sphere transformation necessarily passes through the complex numbers it has not really found its rightful place in modern differential geometry. Of the major classic works on the subject it seems that only the books by Darboux and Blaschke treat it, in fact much of Blaschke's third volume from 1929 is devoted to it. For a modern treatment see Cecil's book [10].

Lie in his thesis, and in the above-mentioned expansion thereof, embarks also on a general theory of contact-transformations, transformations of the cotangent bundle $T^*(\mathbf{C}^n)$ into itself preserving the canonical 1-form (the line-sphere transformation being an example). In the usual coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ on $T^*(\mathbf{C}^n)$, a contact transformation is an isomorphism $(x_i, p_i) \to (X_i, P_i)$ such that $\sum p_i x_i = \sum P_i X_i$. Such transformations appear in the integration of first order partial differential equations

$$F\Big(x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\Big) = 0 \tag{1}$$

In this context consider the Poisson bracket of functions

$$(f, g) = \sum_i \Big(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}\Big) \tag{2}$$

and vector fields

$$X = \sum_j a_j \frac{\partial}{\partial x_j}, \qquad Y = \sum_k b_k \frac{\partial}{\partial x_k} \tag{3}$$

and the Lie bracket

$$[X, Y] = \sum_{jk} \Big(a_j \frac{\partial b_k}{\partial x_j} - b_j \frac{\partial a_k}{\partial x_j}\Big)\frac{\partial}{\partial x_k}. \tag{4}$$

Lie views the Poisson bracket as the effect on f of a vector field X_g associated with g and interprets the Jacobi identity for (f, g) as the fact that the bracket of the operators corresponding to g and h is associated to (g, h):

$$[X_g, X_h] = X_{(g,h)}.$$

Thus the first order differential equation for g

$$(F, g) = 0$$

which enters in Jacobi's work on solving (1) becomes for Lie the equation

$$[X_g, X_F] = 0\,,$$

which leads to a search for an infinitesimal contact transformation leaving invariant the given equation (1). This signals a shift in his work towards transformation groups and differential equations.

After Lie had applied for a professorship in Lund the Norwegian storting approved a Professorship for him in Christiania, July 1, 1872. He continued his work on contact transformations but then in 1873 started on a systematic study of transformation groups. The motivation is the question, stated explicitly in a paper [20] from 1874: "How can the knowledge of a stability group for a differential equation be utilized towards its integration?" A point transformation is said to leave a differential equation stable if it permutes the solutions. Here Lie is of course inspired by Galois theory for algebraic equations which he had heard about already in Sylow's lecture in 1862, and had presumably discussed with Jordan in Paris, who had just then (1869) published his clarification of Galois' theory. In the quoted paper Lie proves the following now famous theorem, stated below.

Consider

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)}$$

and a local one-parameter subgroup ϕ_t of diffeomorphisms of $\mathbf{R}^2$ with induced vector field

$$\Psi_p = \left\{\frac{\partial \phi_t(p)}{\partial t}\right\}_{t=0} = \xi(p)\partial/\partial x + \eta(p)\partial/\partial y\,.$$

Theorem. The transformations ϕ_t leave the equation stable if and only if the vector field $Z = X\partial/\partial x + Y\partial/\partial y$ satisfies

$$[\Psi, Z] = \lambda Z \qquad (\lambda \text{ a function}).$$

In this case $(X\eta - Y\xi)^{-1}$ is an integrating factor to the equation

$$X\,dy - Y\,dx = 0.$$

Example. Consider the differential equation

$$\frac{dy}{dx} = \frac{y + x(x^2+y^2)}{x - y(x^2+y^2)}$$

The equation can be written

$$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x}\frac{dy}{dx}} = x^2 + y^2$$

The left hand side equals $\tan\alpha$ where α is the angle at (x,y) between the integral curve through (x,y) and the radius from (0,0) to (x,y). The right hand side depends only on the distance from the origin so the angle α is constant on the circle with center (0,0) passing through (x,y). Consequently, the rotation group

$$\phi_t : (x,y) \to (x\cos t - y\sin t, x\sin t + y\cos t)$$

for which

$$\Psi = -y\partial/\partial x + x\partial/dy$$

leaves the equation stable. The theorem gives $(x^2+y^2)^{-1}$ as an integrating factor to the original equation.

Another example is the equation

$$\frac{du}{dx} = \left(\frac{d^2v}{dx^2}\right)^2$$

which occurs as an example for different purposes in a paper [17] by Hilbert. Cartan in [6, 7] showed that this equation has the exceptional group G_2 as a stability group (transformations in the independent variable x and dependent variables u and v allowed). See Anderson, Kamran ond Olver [1] for a modern treatment.

Generalizations of the theorem above to systems ([21]) led Lie to develop a theory for r-parameter local groups of transformations in $\mathbf{R}^n$:

$$T_{(t)} : x_i' = f_i(x_1, \ldots, x_n; t_1, \ldots, t_r), \qquad (t) = (t_1, \ldots, t_r). \qquad (5)$$

Here it is assumed that the f_i are analytic functions, depending essentially on all the t_i and that

$$T_{(0)} = I, \qquad T_{(t)} \circ T_{(s)} = T_{(u)}, \qquad (6)$$

where $(u) = (u_1, \ldots, u_r)$ depends analytically on (t) and (s). The group ϕ_t above is a special case.

Lie's big project (expressed in print 1874) was to determine all such transformation groups $T_{(t)}$ and apply the results to the solving of differential equations.

Lie's fundamental results towards solving this problem are as follows:

Generalizing the vector field Ψ above consider the vector fields (infinitesimal transformations)

$$X_k = \sum_i \left(\frac{\partial f_i}{\partial t_k}\right)_{(t)=0} \frac{\partial}{\partial x_i}. \tag{7}$$

As a result of the group property (5) Lie proved the fundamental relation

$$[X_k, X_l] = \sum_p c_{pkl} X_p, \tag{8}$$

where the c_{pkl} are *constants* satisfying

$$c_{pkl} = -c_{plk}, \qquad \sum_q (c_{pkq}c_{qlm} + c_{pmq}c_{qkl} + c_{plq}c_{qmk}) = 0. \tag{9}$$

Conversely, every system of constants c_{pkl} satisfying (9) arises in this way.

This was a major accomplishment and the original proofs were very complicated. Part of the problem is to establish one-parameter subgroups inside the original group (5). These results reduce the internal study of local transformation groups to the study of Lie algebras, that is vector spaces with a rule of composition whose structural constants given by (8) satisfy the relations (9). One can view this assignment as a kind of a non-commutative logarithm.

Looking back one is led to consider this deep and original work as a milestone in the history of mathematics. But while Lie intended this as a tool in the study of differential equations the influence in other fields has been much greater after the general theory of Lie groups had grown into a field on its own, independent of transformation group theory.

While the importance of this work may be obvious to us, being the germ of Lie group theory, this was not so at the time (1873–1876). Very few people took notice and this was a cause of disappointment to Lie. Part of the problem was that Lie was a synthetic geometer at heart whereas this work had to be written in an analytic formulation. This caused problems for both geometers and analysts, and understanding of Lie's work suffered. Lie was aware of this communication problem and tried to cope with it by publishing more and more, faster and faster (in the interest of faster publication he started with G. O. Sars and W. Müller a new journal "Archiv for Mathematik og Naturvidenskab") but these efforts hardly had the intended effect. Thus he turned again to geometric questions in 1876. He did considerable work on minimal surfaces and translation surfaces

$$x = a(t) + A(s), \quad y = b(t) + B(s), \quad z = c(t) + C(s).$$

and later on managed to determine all minimal surfaces which can be realized as translation surfaces in several ways. This turned out to have an interesting connection with earlier work of Abel in 4th degree algebraic curves.

During the years 1873–1876 Lie was completely absorbed in transformation group questions and worked extremely hard. For example he struggled with bare hands and relatively primitive methods at determining all local transformation groups in two variables. Mountains of paper piled up but later Lie accomplished the same with more effective methods. Lie did his classification for two complex variables; the extension to the classification of Lie algebras of vector fields in the real plane has only been completed rather recently, in work [13] of Gonzalez-Lopez, Kamran and Olver. They come up with 28 cases.

Having visited Paris in 1882 and noticing some work of Halphen being included in his own general theories Lie turns again to transformation groups and applications to differential equations, regretting that this work of his in these areas seems overlooked. In a letter to Adolf Mayer 1884 he writes: "If only I knew how to get mathematicians interested in transformation groups and their applications to differential equations. I am certain, absolutely certain, that these theories will some time in the future be recognized as fundamental. When I wish such a recognition sooner, it is partly because then I could accomplish ten times more."

Being aware of Lie's isolation in Christiania, F. Klein and A. Mayer arrange for Klein's student in Leipzig, Friedrich Engel (then 22) to go there to work with Lie. Lie receives him with open arms, and Engel refers later to this year in collaboration with Lie in Christiania as the happiest of his life. Lie decided time was ripe for a full exposition of his theory of transformation groups and Engel was the perfect collaborator. Lie and Engel met every day, in the morning in Engels' apartment, at Lie's in the afternoon. Lie produced a skeleton of each section, discussed contents in detail and left it to Engel to provide this with flesh and blood. The projected work was completed after nine years of labor and appeared in three large volumes between 1888 an 1893.

In 1886 Lie was offered a professorship in Leipzig as a successor to Klein who had been appointed a professor in Göttingen. Klein was instrumental in this appointment of Lie, but Weierstrass (in Berlin) was far from enthusiastic. In Leipzig Lie lectured on his own research but had more duties than in Christiania, and the language was a bit of a problem. He had many students, including some from Paris (Tresse and Vessiot). Of 56 Ph. D.s in Leipzig 1887–1898, 26 wrote their thesis under Lie's supervision.

As is well known from Klein's memoirs, his health suffered through overwork in 1882. In 1889 it became Lie's turn. He suffered constant sleeplessness, light sensitivity and extreme nervousness. He spent several months at a sanatorium in Hanover. The situation improved, and he started on his former work in the summer 1890 and took up again his university lectures. But as Engel describes it: he was a different human being. Touchy, suspicious and made life difficult even for his closest friends. Outside honors and recognition which so long had been wanting did not particularly console him. The illness was diagnosed as Pernicious Anemia. It is caused by the body's inability to absorb B_{12}-vitamin. The same illness hit Hilbert in 1925. Hilbert was lucky because earlier that year Whipple and Robscheit-Robbins had discovered the beneficial effects of raw liver on blood regeneration and this was successfully applied to the treatment of pernicious anemia by G. R. Minot. Hilbert was given this treatment and recovered.

Tragically this cure was not known during Lie's lifetime. He was in addition depressed in Leipzig, missed the magnificent and mountainous nature of his old homeland, his relations with his old friends Klein and Engel deteriorated around 1992–3 (because of scholarship questions mentioned later) in spite of the beneficial influence and assistance they had rendered in Lie's research and in the dissemination thereof. Thus a contemplated book with Engel on "Differential Invariants and Infinite Continuous Groups with Applications to Differential Equations" was never written. The magnanimous Engel attributes this directly to Lie's illness. Also Klein took Lie's public rebuke in [23] philosophically and wrote a strong and successful recommendation of Lie for the first Lobachevsky price 1897. Mathematically this was particularly appropriate because Lie had some years before (1890) solved the so-called Helmholz Space-Problem in terms of non-Euclidean geometry. In modern terms the problem amounts to:

> Let $X = G/H$ be a noncompact homogeneous space for which the group G is simply transitive on the set of all flags. Then X is either a Euclidean or a non-Euclidean space.

Lie first showed that Helmholz' solution was defective and then gave his own proof as an application of his group theory.

In 1892 Lie spent six months in Paris. Élie Cartan gives in [8] a vivid description of his interaction with many young French mathematicians; he would often be seen with them around a table at the Café de la Source on Boulevard Saint-Michel covering the white marble table with formulas. This was also the time when Cartan was writing his magnificent thesis.

In 1896 the Norwegian Storting makes through an initiative from Elling Holst and Bjrnstjerne Bjrnsson offer to Lie of an honorary professorship in Christiania. In 1898 Lie accepts the offer, resigns his position in Leipzig "In grösster Ehrerbietung und in aufrichtiger Dankbarkeit", explaining that he still needs several years to realize his literary plans and that Christiania would probably be more favorable both for his health and his energy. In 1898 he does return to Christiania; carries on some lectures in his apartment but his illness worsened and he died on Feb 18, 1899 at the age of 56.

Lies work has as already mentioned had enormous impact on mathematics and mathematical physics today. Let me comment on this in more detail although I am aware that this summary is very incomplete.

Of Lie's work in geometry his solution of the Helmholz problem was a fine clearcut contribution, his work on minimal surfaces a substantial contribution to a topic which has been in active development even in modern times. His work centering around the line-sphere transformation has however been somewhat bypassed during the years after 1930 when rigorization of differential geometry has taken place. The reason is Lie's unhesitating complexification of some parameters which I explained earlier. However, as I also explained there is an important real side to the story and applications of this have been actively explored during the last 15 years or so. It is therefore quite likely that this work of Lie's will be undergoing a reexamination.

His work on transformation groups has by far been the most influential.

The purely algebraic problem of classifying simple Lie algebras was taken up by Killing, and corrected and completed by Élie Cartan. The subject blossomed into a very complete theory of global Lie groups which in its numerous ramifications like representation theory, infinite-dimensional Lie algebras, algebraic groups, p-adic groups, etc. is intensively studied today. In addition, the influence of Lie group theory on other mathematical disciplines has become more and more pervasive; here I mention differential geometry, harmonic analysis, integral geometry, topology, combinatorics, number theory and finite group theory and of course, mathematical physics.

Lie's work on partial differential equations calls for a more extended comment.

Lie's result from 1874 for ordinary differential equations which I stated before was a profound discovery showing that many existing procedures for solving such equations could be subsumed under one principle, namely the stability of the equation under a one-parameter group of symmetries. This suggested the program of finding an analog of Galois theory for differential equations. Since the solutions to differential quations are functions, not numbers, there are two natural directions one can take for such an analogue.

Analytic viewpoint (Lie 1871–1874) For a system of differential equations consider the group of diffeomorphisms of the space leaving the differential equation system stable.

Algebraic Viewpoint (Picard 1883, Vessiot 1891) For a given differential equation consider the automorphism of the field generated by the solutions, fixing the coefficient field.

In this latter viewpoint one introduces the so-called differential Galois group (since the automorphisms are supposed to commute with differentiation). The solvability of this group is then a necessary and sufficient for the equation to be solvable by quadratures. This has developed into the subject differential algebra.

Lie's program (the first viewpoint indicated above) was continued by a number of mathematicians but partial differential equation theory has taken a different direction under the influence of functional analysis. There is not so much emphasis of determining the solutions but more interest in existence, uniqueness, smoothness properties etc. Thus one does not generally find Lie quoted in recent authorative works on partial differential equations. However on the side, Lie 's program to study individual differential equations by means of their symmetry group is very much alive and actively pursued in many quarters. For example the Korteweg-deVries equation $u_t + u_{xxx} + u\,u_x = 0$ has a symmetry algebra spanned by four vector fields, and this gives certain insight in properties of the solutions [25]. Finding symmetry properties of such given equations sometimes reduces to rather mechanical computations and certain computer programs have been developed for this task. The books [3], [25] provide a good introduction to this field. See also Ibragimov's lectures at this conference.

It is also possible to turn Lie's program around and take the Lie transformation group as the given object and study differential equations and differential operators invariant under that given group. With global Lie group theory so well developed this gives rise to a multitude of natural problems, see e.g. [15, 16].

Lie was a patriot and loved the mountainous beauty of Norway. It is good to recall that his country treated him with understanding and generosity, extending also to his widow after his premature death. The magnificent edition of his Collected Works, with all his papers completely reprinted and furnished with thousands of pages of commentaries by Engel and Heegaard, financed jointly by the Academies in Oslo and in Leipzig, is also a testimony to this generosity.

Lie had high standards and was often, particularly after his mentioned illness, critical of other mathematicians. Mathematicians are often poor historians and do not always attribute accurately results they use to their original sources. Lie was particularly sensitive to such situations and used the third volume of [23] to settle accounts. Here one finds examples of tactless overreaction as well as examples of justified criticism. While specific instances of this could easily be quoted and analysed it is hard to do so in a small space without creating an impression which would be misleading. Lie was also rather possessive of his own work, sometimes unwilling in his exposition to adopt simplifications published by others and yet eager for recognition of it by others. After only three years 1873–76 of his work on transformation groups he is complaining of lack of interest of the mathematical public and as a result turns to other topics (In comparison one might mention Élie Cartan who worked on Lie algebras for 25 years without much participation from anyone; it did not seem to bother him). But if someone decided to publish research in transformation group theory then that person was well-advised to show his familiarity with Lie's papers. Even a colleague like Friedrich Schur, who made substantial improvements in the proofs of the fundamental theorems was thus taken to task in print. The mathematician whose work always met with Lie's approval and admiration was Élie Cartan. Let me therefore conclude with Cartan's words spoken on the centenary of Lie's birth (cf. [8]):

> "Sophus Lie was of tall stature and had the classic Nordic appearance. A full blond beard framed his face and his gray-blue eyes sparkled behind his eyeglasses. He gave the impression of unusual physical strength. One always immediately felt at ease with him, certain beforehand of his sincerity and his loyalty. He was not afraid to admit his ignorance of branches of mathematics unfamiliar to him, which nevertheless did not keep him from being aware of his own worth......Posterity will see in him the genius who created the theory of transformation groups, and we French shall never be able to forget the ties which bind us to him and which make his memory dear to us".

References

[1] Anderson, I. M., Kamran, N., and Olver, P. J., Internal, external and generalized symmetries. *Advan. Math.* 100 (1993), 53–100.

[2] Blaschke, W., Vorlesungen über Differential Geometrie. Vol. I–III. Springer, Berlin, 1929.

[3] Bluman, G. W., and Kumei, S., Symmetries and Differential Equations. Springer-Verlag, New York, 1989.

[4] Bourbaki, N., Élements de mathématiques. FaccXXXVII. *Groupes et Algèbres de Lie (Ch. II–III)*. Hermann, Paris, 1972.

[5] Cartan, É., Les systèmes de Pfaff a cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Ec. Normale* **27** (1910), 109–192.

[6] Cartan, É., Sur l'equivalence absolue de certains systèmes d'équations différentiels et sur certaines familles de courbes. *Bull. Soc. Math. France* **42** (1914), 12–48.

[7] Cartan, É., Sur l'intégration de certains systèmes indéterminés d'équations diffrentielles. *J. Reine Angew. Math.* **145** (1915), 86–91.

[8] Cartan, É., A Centenary: Sophus Lie. In "Great Currents of Mathematical Thought " F. Le Lionnais (Ed). Dover, New York, 1971, 262–267.

[9] Cartan, É., Oevres Complètes, Gauthier-Villars, Paris 1952.

[10] Cecil, T. E., Lie Sphere Geometry. Springer-Verlag, New York, 1992.

[11] Darboux, G., Théorie Génerale des Surfaces, Vol. I–IV. Gauthier Villars, Paris 1887-1896.

[12] Engel, F., Sophus Lie (obituary). *Jber. Deutsch. Math.-Verein* **8** (1900), 30–46.

[13] González-Lopez, A., Kamran, N., and Olver, P. J., Lie algebras of vector fields in the plane. *Proc. London Math. Soc.* (to appear).

[14] Hawkins, T., Line geometry, differential equations, and the birth of Lie's theory of groups. In "The History of Modern Mathematics" D. E. Rowe and J. McCleary, eds. Academic Press, New York, 1989.

[15] Helgason, S., Differential operators on homogeneous spaces. *Acta Math.* **102** (1959), 239–299.

[16] Helgason, S., Invariant differential equations on homogeneous manifolds. *Bull. Amer. Math. Soc.* **83** (1977), 751–774.

[17] Hilbert, D., Über den Begriff der Klasse von Differentialgleichungen. *Math. Ann.* **73** (1912), 95–108.

[18] Klein, F., Vorlesungen über Höhere Geometrie, Springer, Berlin 1926.

[19] Klein, F., Gesammelte Mathematische Abhandlungen, Vol. I–III, Berlin 1921–1923.

[20] Lie, S., Zur Theorie des Integrabilitetsfaktors, Christiania Forh. 1874, 242–254; reprint in [22], Vol. III, no. XIII, pp. 176–187.

[21] Lie, S., Verallgemeinerung und neue Verwertung der Jacobischen Multiplikatortheorie, Christiania Forh. 1874, 254–274.

[22] Lie, S., Gesammelte Abhandlungen, 7 Vols. Teubner, Leipzig, 1922–1960.

[23] Lie, S., and Engel, F., Theorie de Transformationsgruppen, 3 vols, Teubner, Leipzig 1888–1893.

[24] Mostow, G. D., Continuous groups, Encyclopaedia Britannica, 1967.

[25] Olver, P., Applications of Lie Groups to Differential Equations. Springer Verlag, New York 1986.

[26] Pommaret, J. F., Differential Galois Theory, Gordon and Breach, New York, 1983.