2. Classical and quantum Olshanetsky-Perelomov systems for finite Coxeter groups

2.1. **The rational quantum Calogero-Moser system.** Consider the differential operator

$$H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - c(c+1)\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}.$$

This is the quantum Hamiltonian for a system of n particles on the line of unit mass and the interaction potential (between particle 1 and 2) $c(c+1)/(x_1 - x_2)^2$. This system is called *the rational quantum Calogero-Moser system.*

It turns out that the rational quantum Calogero-Moser system is completely integrable. Namely, we have the following theorem.

Theorem 2.1. *There exist differential operators L_j with rational coefficients of the form*

$$L_j = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^j + \text{ lower order terms}, \quad j = 1, \ldots, n,$$

which are invariant under the symmetric group $\mathfrak{S}_n$, homogeneous of degree $-j$, and such that $L_2 = H$ and $[L_j, L_k] = 0, \forall j, k = 1, \ldots, n$.

We will prove this theorem later.

Remark 2.2. $L_1 = \sum_i \dfrac{\partial}{\partial x_i}$.

2.2. **Complex reflection groups.** Theorem 2.1 can be generalized to the case of any finite Coxeter group. To describe this generalization, let us recall the basic theory of finite Coxeter groups and, more generally, complex reflection groups.

Let $\mathfrak{h}$ be a finite-dimensional complex vector space. We say that a semisimple element $s \in \mathrm{GL}(\mathfrak{h})$ is a *(complex) reflection* if $\operatorname{rank}(1 - s) = 1$. This means that s is conjugate to the diagonal matrix $\mathrm{diag}(\lambda, 1, \ldots, 1)$ where $\lambda \neq 1$.

Now assume $\mathfrak{h}$ carries a nondegenerate inner product $(\cdot, \cdot)$. We say that a semisimple element $s \in \mathrm{O}(\mathfrak{h})$ is a *real reflection* if $\operatorname{rank}(1 - s) = 1$; equivalently, s is conjugate to $\mathrm{diag}(-1, 1, \ldots, 1)$.

Now let $G \subset \mathrm{GL}(\mathfrak{h})$ be a finite subgroup.

Definition 2.3.
- (i) We say that G is a *complex reflection group* if it is generated by complex reflections.
- (ii) If $\mathfrak{h}$ carries an inner product, then a finite subgroup $G \subset \mathrm{O}(\mathfrak{h})$ is a *real reflection group* (or a *finite Coxeter group*) if G is generated by real reflections.

For the complex reflection groups, we have the following important theorem.

Theorem 2.4 (The Chevalley-Shepard-Todd theorem, [Che])**.** *A finite subgroup G of* $\mathrm{GL}(\mathfrak{h})$ *is a complex reflection group if and only if the algebra $(S\mathfrak{h})^G$ is a polynomial (i.e., free) algebra.*

By the Chevalley-Shepard-Todd theorem, the algebra $(S\mathfrak{h})^G$ has algebraically independent generators P_i, homogeneous of some degrees d_i for $i = 1, \ldots, \dim \mathfrak{h}$. The numbers d_i are uniquely determined, and are called *the degrees* of G.

Example 2.5. If $G = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$ (the space of vectors in $\mathbb{C}^n$ with zero sum of coordinates), then one can take $P_i(p_1, \ldots, p_n) = p_1^{i+1} + \cdots + p_n^{i+1}$, $i = 1, \ldots, n-1$ (where $\sum_i p_i = 0$).

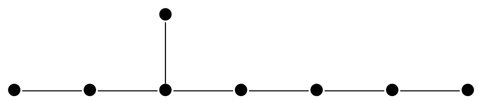
2.3. **Parabolic subgroups.** Let $G \subset \mathrm{GL}(\mathfrak{h})$ be a finite subgroup.

Definition 2.6. A parabolic subgroup of G is the stabilizer G_a of a point $a \in \mathfrak{h}$.

Note that by Chevalley's theorem, a parabolic subgroup of a complex (respectively, real) reflection group is itself a complex (respectively, real) reflection group.

Also, if W is a real reflection group, then it can be shown that a subgroup $W' \subset W$ is parabolic if and only if it is conjugate to a subgroup generated by a subset of simple reflections of W. In this case, the rank of W', i.e. the number of generating simple reflections, equals the codimension of the space $\mathfrak{h}^{W'}$.

Example 2.7. Consider the Coxeter group of type E_8. It has the Dynkin diagram:



The parabolic subgroups will be Coxeter groups whose Dynkin diagrams are obtained by deleting vertices from the above graph. In particular, the maximal parabolic subgroups are $D_7, A_7, A_1 \times A_6, A_2 \times A_1 \times A_4, A_4 \times A_3, D_5 \times A_2, E_6 \times A_1, E_7$.

Suppose $G' \subset G$ is a parabolic subgroup, and $b \in \mathfrak{h}$ is such that $G_b = G'$. In this case, we have a natural G'-invariant decomposition $\mathfrak{h} = \mathfrak{h}^{G'} \oplus (\mathfrak{h}^{*G'})^{\perp}$, and $b \in \mathfrak{h}^{G'}$. Thus we have a nonempty open set $\mathfrak{h}^{G'}_{\mathrm{reg}}$ of all $a \in \mathfrak{h}^{G'}$ for which $G_a = G'$; this set is nonempty because it contains b. We also have a G'-invariant decomposition $\mathfrak{h}^* = \mathfrak{h}^{*G'} \oplus (\mathfrak{h}^{G'})^{\perp}$, and we can define the open set $\mathfrak{h}^{*G'}_{\mathrm{reg}}$ of all $\lambda \in \mathfrak{h}^{G'}$ for which $G_\lambda = G'$. It is clear that this set is nonempty. This implies, in particular, that one can make an alternative definition of a parabolic subgroup of G as the stabilizer of a point in $\mathfrak{h}^*$.

2.4. **Olshanetsky-Perelomov operators.** Let $s \subset \mathrm{GL}(\mathfrak{h})$ be a complex reflection. Denote by $\alpha_s \in \mathfrak{h}^*$ an eigenvector in $\mathfrak{h}^*$ of s with nontrivial eigenvalue.

Let $W \subset \mathrm{O}(\mathfrak{h})$ be a real reflection group and $\mathcal{S} \subset W$ the set of reflections. Clearly, W acts on $\mathcal{S}$ by conjugation. Let $c : \mathcal{S} \to \mathbb{C}$ be a conjugation invariant function.

Definition 2.8. [OP] The quantum Olshanetsky-Perelomov Hamiltonian attached to W is the second order differential operator

$$H := \Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2},$$

where $\Delta_{\mathfrak{h}}$ is the Laplace operator on $\mathfrak{h}$.

Here we use the inner product on $\mathfrak{h}^*$ which is dual to the inner product on $\mathfrak{h}$.

Let us assume that $\mathfrak{h}$ is an irreducible representation of W (i.e. W is an irreducible finite Coxeter group, and $\mathfrak{h}$ is its reflection representation.) In this case, we can take $P_1(\mathbf{p}) = \mathbf{p}^2$.

Theorem 2.9. *The system defined by the Olshanetsky-Perelomov operator H is completely integrable. Namely, there exist differential operators L_j on $\mathfrak{h}$ with rational coefficients and symbols P_j, such that L_j are homogeneous (of degree $-d_j$), $L_1 = H$, and $[L_j, L_k] = 0$, $\forall j, k$.*

This theorem is obviously a generalization of Theorem 2.1 about $W = \mathfrak{S}_n$.

To prove Theorem 2.9, one needs to develop the theory of Dunkl operators.

Remark 2.10. 1. We will show later that the operators L_j are unique.

2. Theorem 2.9 for classical root systems was proved by Olshanetsky and Perelomov (see [OP]), following earlier work of Calogero, Sutherland, and Moser in type A. For a general Weyl group, this theorem (in fact, its stronger trigonometric version) was proved by analytic methods in the series of papers [HO],[He3],[Op3],[Op4]. A few years later, a simple algebraic proof using Dunkl operators, which works for any finite Coxeter group, was found by Heckman, [He1]; this is the proof we will give below.

For the trigonometric version, Heckman also gave an algebraic proof in [He2], which used non-commuting trigonometric counterparts of Dunkl operators. This proof was later improved by Cherednik ([Ch1]), who defined commuting (although not Weyl group invariant) versions of Heckman's trigonometric Dunkl operators, now called Dunkl-Cherednik operators.

2.5. **Dunkl operators.** Let $G \subset \mathrm{GL}(\mathfrak{h})$ be a finite subgroup. Let $\mathcal{S}$ be the set of reflections in G. For any reflection $s \in \mathcal{S}$, let λ_s be the eigenvalue of s on $\alpha_s \in \mathfrak{h}^*$ (i.e. $s\alpha_s = \lambda_s \alpha_s$), and let $\alpha_s^\vee \in \mathfrak{h}$ be an eigenvector such that $s\alpha_s^\vee = \lambda_s^{-1}\alpha_s^\vee$. We normalize them in such a way that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

Let $c : \mathcal{S} \to \mathbb{C}$ be a function invariant with respect to conjugation. Let $a \in \mathfrak{h}$.

The following definition was made by Dunkl for real reflection groups, and by Dunkl and Opdam for complex reflection groups.

Definition 2.11. The Dunkl operator $D_a = D_a(c)$ on $\mathbb{C}(\mathfrak{h})$ is defined by the formula

$$D_a = D_a(c) := \partial_a - \sum_{s \in \mathcal{S}} \frac{2c_s \alpha_s(a)}{(1-\lambda_s)\alpha_s}(1-s).$$

Clearly, $D_a \in \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$, where $\mathfrak{h}_{\mathrm{reg}}$ is the set of regular points of $\mathfrak{h}$ (i.e. not preserved by any reflection), and $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ denotes the algebra of differential operators on $\mathfrak{h}_{\mathrm{reg}}$.

Example 2.12. Let $G = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. Then there is only one Dunkl operator up to scaling, and it equals to

$$D = \partial_x - \frac{c}{x}(1-s),$$

where the operator s is given by the formula $(sf)(x) = f(-x)$.

Remark 2.13. The Dunkl operators D_a map the space of polynomials $\mathbb{C}[\mathfrak{h}]$ to itself.

Proposition 2.14. (i) *For any $x \in \mathfrak{h}^*$, one has*

$$[D_a, x] = (a, x) - \sum_{s \in \mathcal{S}} c_s (a, \alpha_s)(x, \alpha_s^\vee) s.$$

(ii) *If $g \in G$ then $g D_a g^{-1} = D_{ga}$.*

Proof. (i) The proof follows immediately from the identity

$$x - sx = \frac{1-\lambda_s}{2}(x, \alpha_s^\vee)\alpha_s.$$

(ii) The identity is obvious from the invariance of the function c. □

The main result about Dunkl operators, on which all their applications are based, is the following theorem.

Theorem 2.15 (C. Dunkl, [Du1]). *The Dunkl operators commute:*

$$[D_a, D_b] = 0 \text{ for any } a, b \in \mathfrak{h}.$$

Proof. Let $x \in \mathfrak{h}^*$. We have

$$[[D_a, D_b], x] = [[D_a, x], D_b] - [[D_b, x], D_a].$$

Now, using Proposition 2.14, we obtain:

$$\begin{aligned} [[D_a, x], D_b] &= -[\sum_{s \in \mathcal{S}} c_s(a, \alpha_s)(x, \alpha_s^\vee)s, D_b] \\ &= -\sum_{s \in \mathcal{S}} c_s(a, \alpha_s)(x, \alpha_s^\vee)(b, \alpha_s) s D_{\alpha_s^\vee} \cdot \frac{1 - \lambda_s^{-1}}{2}. \end{aligned}$$

Since a and b occur symmetrically, we obtain that $[[D_a, D_b], x] = 0$. This means that for any $f \in \mathbb{C}[\mathfrak{h}]$, $[D_a, D_b]f = f[D_a, D_b]1 = 0$. So for $f, g \in \mathbb{C}[\mathfrak{h}]$, $g \cdot [D_a, D_b]\frac{f}{g} = [D_a, D_b]f = 0$. Thus $[D_a, D_b]\frac{f}{g} = 0$ which implies $[D_a, D_b] = 0$ in the algebra $\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}})$ (since this algebra acts faithfully on $\mathbb{C}(\mathfrak{h})$). $\square$

2.6. **Proof of Theorem 2.9.** For any element $B \in \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}})$, define $m(B)$ to be the differential operator $\mathbb{C}(\mathfrak{h})^W \to \mathbb{C}(\mathfrak{h})$, defined by B. That is, if $B = \sum_{g \in W} B_g g$, $B_g \in \mathcal{D}(\mathfrak{h}_{\text{reg}})$, then $m(B) = \sum_{g \in W} B_g$. It is clear that if B is W-invariant, then $\forall A \in \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}})$,

$$m(AB) = m(A)m(B).$$

Proposition 2.16 ([Du1], [He1]). *Let $\{y_1, \ldots, y_r\}$ be an orthonormal basis of $\mathfrak{h}$. Then we have*

$$m(\sum_{i=1}^r D_{y_i}^2) = \overline{H},$$

where $\overline{H} = \Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^\vee}$.

Proof. For any $y \in \mathfrak{h}$, we have $m(D_y^2) = m(D_y \partial_y)$. A simple computation shows that

$$\begin{aligned} D_y \partial_y &= \partial_y^2 - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(y)}{\alpha_s}(1 - s)\partial_y \\ &= \partial_y^2 - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(y)}{\alpha_s}(\partial_y(1 - s) + \alpha_s(y)\partial_{\alpha_s^\vee} s). \end{aligned}$$

This means that

$$m(D_y^2) = \partial_y^2 - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(y)^2}{\alpha_s} \partial_{\alpha_s^\vee}.$$

So we get

$$m(\sum_{i=1}^r D_{y_i}^2) = \sum_{i=1}^r \partial_{y_i}^2 - \sum_{s\in\mathcal{S}} c_s \sum_{i=1}^r \frac{\alpha_s(y_i)^2}{\alpha_s}\partial_{\alpha_s^\vee} = \overline{H},$$

since $\sum_{i=1}^r \alpha_s(y_i)^2 = (\alpha_s, \alpha_s)$. □

Recall that by the Chevalley-Shepard-Todd theorem, the algebra $(S\mathfrak{h})^W$ is free. Let $P_1 = \mathbf{p}^2, P_2, \ldots, P_r$ be homogeneous generators of $(S\mathfrak{h})^W$.

Corollary 2.17. *The differential operators $\overline{L}_j = m(P_j(D_{y_1}, \ldots, D_{y_r}))$ are pairwise commutative, have symbols P_j, homogeneity degree $-d_j$, and $\overline{L}_1 = \overline{H}$.*

Proof. Since Dunkl operators commute, the operators L_j are well defined. Since $m(AB) = m(A)m(B)$ when B is invariant, the operators L_j are pairwise commutative. The rest is clear. □

Now to prove Theorem 2.9, we will show that the operators H and $\overline{H}$ are conjugate to each other by a certain function; this will complete the proof.

Proposition 2.18. *Let $\delta_c(\mathbf{x}) := \prod_{s\in\mathcal{S}} \alpha_s(\mathbf{x})^{c_s}$. Then we have*

$$\delta_c^{-1} \circ \overline{H} \circ \delta_c = H.$$

Remark 2.19. The function $\delta_c(\mathbf{x})$ is not rational. It is a multivalued analytic function. Nevertheless, it is easy to see that for any differential operator L with rational coefficients, $\delta_c^{-1} \circ L \circ \delta_c$ also has rational coefficients.

Proof of Proposition 2.18. We have

$$\sum_{i=1}^r \partial_{y_i}(\log \delta_c)\partial_{y_i} = \sum_{s\in\mathcal{S}} \frac{c_s(\alpha_s, \alpha_s)}{2\alpha_s}\partial_{\alpha_s^\vee}.$$

Therefore, we have

$$\delta_c \circ H \circ \delta_c^{-1} = \Delta_\mathfrak{h} - \sum_{s\in\mathcal{S}} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s}\partial_{\alpha_s^\vee} + U,$$

where

$$U = \delta_c(\Delta_\mathfrak{h}\delta_c^{-1}) - \sum_{s\in\mathcal{S}} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2}.$$

Let us compute U. We have

$$\delta_c(\Delta_\mathfrak{h}\delta_c^{-1}) = \sum_{s\in\mathcal{S}} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2} + \sum_{s\neq u\in\mathcal{S}} \frac{c_s c_u(\alpha_s, \alpha_u)}{\alpha_s\alpha_u}.$$

We claim that the last sum Σ is actually zero. Indeed, this sum is invariant under the Coxeter group, so $\prod_{s\in\mathcal{S}} \alpha_s \cdot \Sigma$ is a regular anti-invariant function of degree $|\mathcal{S}| - 2$. But the smallest degree of a nonzero anti-invariant is $|\mathcal{S}|$, so $\Sigma = 0$, $U = 0$, and we are done (Proposition 2.18 and Theorem 2.9 are proved). □

Remark 2.20. A similar method works for any complex reflection group G. Namely, the operators $L_i = m(P_i(D_{y_1}, \ldots, D_{y_r}))$ form a quantum integrable system. However, if G is not a real reflection group, this system does not have a quadratic Hamiltonian in momentum variables (so it does not have a physical meaning).

2.7. **Uniqueness of the operators L_j.**

Proposition 2.21. *The operators L_j are unique.*

Proof. Assume that we have two choices for L_j: L_j and L_j'. Denote $L_j - L_j'$ by M.

Assume $M \neq 0$. We have

(i) M is a differential operator on $\mathfrak{h}$ with rational coefficients, of order smaller than d_j and homogeneity degree $-d_j$;

(ii) $[M, H] = 0$.

Let M_0 be the symbol of M. Then M_0 is a polynomial of $\mathbf{p} \in \mathfrak{h}^*$ with coefficients in $\mathbb{C}(\mathfrak{h})$. We have, from (ii),

$$\{M_0, \mathbf{p}^2\} = 0, \quad \forall \mathbf{p} \in \mathfrak{h}^*,$$

and from (i) we see that the coefficients of M_0 are not polynomial (as they have negative degree).

However, we have the following lemma.

Lemma 2.22. *Let $\mathfrak{h}$ be a finite dimensional vector space. Let $\psi : (\mathbf{x}, \mathbf{p}) \mapsto \psi(\mathbf{x}, \mathbf{p})$ be a rational function on $\mathfrak{h} \oplus \mathfrak{h}^*$ which is a polynomial in $\mathbf{p} \in \mathfrak{h}^*$. Let $f : \mathfrak{h}^* \to \mathbb{C}$ be a polynomial such that the differentials $\mathrm{d}f(\mathbf{p})$ for $\mathbf{p} \in \mathfrak{h}^*$ span $\mathfrak{h}$ (e.g., $f(\mathbf{p}) = \mathbf{p}^2$). Suppose that the Poisson bracket of f and ψ vanishes: $\{\psi, f\} = 0$. Then ψ is a polynomial.*

Proof. (R. Raj) Let $\mathfrak{Z} \subset \mathfrak{h}$ be the pole divisor of ψ. Let $\mathbf{x}_0 \in \mathfrak{h}$ be a generic point in $\mathfrak{Z}$. Then ψ^{-1} is regular and vanishes at $(\mathbf{x}_0, \mathbf{p})$ for generic $\mathbf{p} \in \mathfrak{h}^*$. Also from $\{\psi^{-1}, f\} = 0$, we have ψ^{-1} vanishes along the entire flowline of the Hamiltonian flow defined by f and starting at $\mathbf{x}_0$. This flowline is defined by the formula

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathrm{d}f(\mathbf{p}), \quad \mathbf{p}(t) = \mathbf{p},$$

and it must be contained in the pole divisor of ψ near $\mathbf{x}_0$. This implies that $\mathrm{d}f(\mathbf{p})$ must be in $T_{\mathbf{x}_0}\mathfrak{Z}$ for almost every, hence for every $\mathbf{p} \in \mathfrak{h}^*$. This is a contradiction with the assumption on f, which implies that in fact ψ has no poles. □

□

2.8. **Classical Dunkl operators and Olshanetsky-Perelomov Hamiltonians.** We continue to use the notations in Section 2.4.

Definition 2.23. The classical Olshanetsky-Perelomov Hamiltonian corresponding to W is the following classical Hamiltonian on $\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^* = T^*\mathfrak{h}_{\mathrm{reg}}$:

$$H_0(\mathbf{x}, \mathbf{p}) = \mathbf{p}^2 - \sum_{s \in \mathcal{S}} \frac{c_s^2(\alpha_s, \alpha_s)}{\alpha_s^2(\mathbf{x})}.$$

Theorem 2.24 ([OP],[HO, He3, Op3, Op4],[He1])**.** *The Hamiltonian H_0 defines a classical integrable system. Namely, there exist unique regular functions L_j^0 on $\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*$, where highest terms in $\mathbf{p}$ are P_j, such that L_j^0 are homogeneous of degree $-d_j$ (under $\mathbf{x} \mapsto \lambda\mathbf{x}, \mathbf{x} \in \mathfrak{h}^*, \mathbf{p} \mapsto \lambda^{-1}\mathbf{p}, \mathbf{p} \in \mathfrak{h}$), and such that $L_1^0 = H_0$ and $\{L_j^0, L_k^0\} = 0, \forall j, k$.*

Proof. The proof is given in the next subsection. □

Example 2.25. Let $W = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$. Then

$$H_0 = \sum_{i=1}^{n} p_i^2 - c^2 \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \quad \text{(the classical Calogero-Moser Hamiltonian).}$$

So the theorem says that there are functions $L_j^0, j = 1, \dots, n-1$,

$$L_j^0 = \sum_i p_i^{j+1} + \text{ lower terms,}$$

homogeneous of degree zero, such that $L_1^0 = H_0$ and $\{L_j^0, L_k^0\} = 0$.

2.9. **Rees algebras.** Let $\overline{A}$ be a filtered algebra over a field k: $k = F^0\overline{A} \subset F^1\overline{A} \subset \cdots$, $\cup_i F^i\overline{A} = \overline{A}$. Then the *Rees algebra* $A = \mathrm{Rees}(\overline{A})$ is defined by the formula $A = \oplus_{n=0}^{\infty} F^n\overline{A}$. This is an algebra over $k[\hbar]$, where $\hbar$ is the element 1 of the summand $F^1\overline{A}$.

2.10. **Proof of Theorem 2.24.** The proof of Theorem 2.24 is similar to the proof of its quantum analog. Namely, to construct the functions L_j^0, we need to introduce classical Dunkl operators. To do so, we introduce a parameter $\hbar$ (Planck's constant) and define Dunkl operators $D_a(\hbar) = D_a(c, \hbar)$ with $\hbar$:

$$D_a(c, \hbar) = \hbar D_a(c/\hbar) = \hbar\partial_a - \sum_{s \in \mathcal{S}} \frac{2c_s\alpha_s(a)}{(1-\lambda_s)\alpha_s}(1-s), \text{ where } a \in \mathfrak{h}.$$

These operators can be regarded as elements of the Rees algebra $A = \mathrm{Rees}(\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$, where the filtration is by order of differential operators (and W sits in degree 0). Reducing these operators modulo $\hbar$, we get classical Dunkl operators $D_a^0(c) \in A_0 := A/\hbar A = \mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. They are given by the formula

$$D_a^0(c) = p_a - \sum_{s \in \mathcal{S}} \frac{2c_s\alpha_s(a)}{(1-\lambda_s)\alpha_s}(1-s),$$

where p_a is the classical momentum (the linear function on $\mathfrak{h}^*$ corresponding to $a \in \mathfrak{h}$).

It follows from the commutativity of the quantum Dunkl operators $D_a(c)$ that the Dunkl operators $D_a(c, \hbar)$ commute. Hence, so do the classical Dunkl operators D_a^0:

$$[D_a^0, D_b^0] = 0.$$

We also have the following analog of Proposition 2.14:

Proposition 2.26. (i) *For any $x \in \mathfrak{h}^*$, one has*

$$[D_a^0, x] = -\sum_{s \in \mathcal{S}} c_s(a, \alpha_s)(x, \alpha_s^\vee)s.$$

(ii) *If $g \in W$ then $gD_a^0g^{-1} = D_{ga}^0$.*

Now let us construct the classical Olshanetsky-Perelomov Hamiltonians. As in the quantum case, we have the operation $m(\cdot)$, which is given by the formula $\sum_{g \in W} B_g \cdot g \mapsto \sum B_g$, $B \in \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. We define the Hamiltonian

$$\overline{H}_0 := m(\sum_{i=1}^{r}(D_{y_i}^0)^2).$$

By taking the limit of quantum situation, we find

$$\overline{H}_0 = \mathbf{p}^2 - \sum_{s\in\mathcal{S}} \frac{c_s(\alpha_s,\alpha_s)}{\alpha_s(\mathbf{x})} p_{\alpha_s^\vee}.$$

Unfortunately, this is no longer conjugate to H_0. However, consider the (outer) automorphism θ_c of the algebra $\mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$ defined by the formulas

$$\theta_c(x) = x,\ \theta_c(s) = s,\ \theta_c(p_a) = p_a + \partial_a \log \delta_c,$$

for $x \in \mathfrak{h}^*$, $a \in \mathfrak{h}$, $s \in W$. It is easy to see that if $b_0 \in A_0$ and $b \in A$ is a deformation of b_0 then $\theta_c(b_0) = \lim_{\hbar\to 0} \delta_{c/\hbar}^{-1} b \delta_{c/\hbar}$. Therefore, taking the limit $\hbar \to 0$ in Proposition 2.16, we find that $H_0 = \theta_c(\overline{H}_0)$.

Now set $L_j^0 = m(\theta_c(P_j(D_{y_1}^0, \ldots, D_{y_r}^0)))$. These functions are well defined since D_a^0 commute, are homogeneous of degree zero, and $L_1^0 = H_0$.

Moreover, we can define the operators $L_j(\hbar)$ in $\mathrm{Rees}(\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W)$ in the same way as L_j, but using the Dunkl operators $D_{y_i}(\hbar)$ instead of D_{y_i}. Then $[L_j(\hbar), L_k(\hbar)] = 0$, and $L_j(\hbar)|_{\hbar=0} = L_j^0$. This implies that L_j^0 Poisson commute: $\{L_j^0, L_k^0\} = 0$.

Theorem 2.24 is proved.

Remark 2.27. As in the quantum situation, Theorem 2.24 can be generalized to complex reflection groups, giving integrable systems with Hamiltonians which are non-quadratic in momentum variables.

2.11. **Notes.** Section 2.1 follows Section 5.4 of [E4]; the definition of complex reflection groups and their basic properties can be found in [GM]; the definition of parabolic subgroups and the notations are borrowed from Section 3.1 of [BE]; the remaining parts of this section follow Section 6 of [E4].

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