1. Introduction

Double affine Hecke algebras, also called Cherednik algebras, were introduced by Cherednik in 1993 as a tool in his proof of Macdonald's conjectures about orthogonal polynomials for root systems. Since then, it has been realized that Cherednik algebras are of great independent interest; they appeared in many different mathematical contexts and found numerous applications.

The present notes are based on a course on Cherednik algebras given by the first author at MIT in the Fall of 2009. Their goal is to give an introduction to Cherednik algebras, and to review the web of connections between them and other mathematical objects. For this reason, the notes consist of many parts that are relatively independent of each other. Also, to keep the notes within the bounds of a one-semester course, we had to limit the discussion of many important topics to a very brief outline, or to skip them altogether. For a more in-depth discussion of Cherednik algebras, we refer the reader to research articles dedicated to this subject.

The notes do not contain any original material. In each section, the sources of the exposition are listed in the notes at the end of the section.

The organization of the notes is as follows.

In Section 2, we define the classical and quantum Calogero-Moser systems, and their analogs for any Coxeter groups introduced by Olshanetsky and Perelomov. Then we introduce Dunkl operators, prove the fundamental result of their commutativity, and use them to establish integrability of the Calogero-Moser and Olshanetsky-Perelomov systems. We also prove the uniqueness of the first integrals for these systems.

In Section 3, we conceptualize the commutation relations between Dunkl operators and coordinate operators by introducing the main object of these notes - the rational Cherednik algebra. We develop the basic theory of rational Cherednik algebras (proving the PBW theorem), and then pass to the representation theory of rational Cherednik algebras, more precisely, study the structure of category $\mathcal{O}$. After developing the basic theory (parallel to the case of semisimple Lie algebras), we completely work out the representations in the rank 1 case, and prove a number of results about finite dimensional representations and about representations of the rational Cherednik algebra attached to the symmetric group.

In Section 4, we evaluate the Macdonald-Mehta integral, and then use it to find the supports of irrieducible modules over the rational Cherednik algebras with the trivial lowest weight, in particular giving a simple proof of the theorem of Varagnolo and Vasserot, classifying such representations which are finite dimensional.

In Section 5, we describe the theory of parabolic induction and restriction functors for rational Cherednik algebras, developed in [BE], and give some applications of this theory, such as the description of the category of Whittaker modules and of possible supports of modules lying in category $\mathcal{O}$.

In Section 6, we define Hecke algebras of complex reflection groups, and the Knizhnik-Zamolodchikov (KZ) functor from the category $\mathcal{O}$ of a rational Cherednik algebra to the category of finite dimensional representations of the corresponding Hecke algebra. We use this functor to prove the formal flatness of Hecke algebras of complex reflection groups (a theorem of Broué, Malle, and Rouquier), and state the theorem of Ginzburg-Guay-Opdam-Rouquier that the KZ functor is an equivalence from the category $\mathcal{O}$ modulo its torsion part to the category of representations of the Hecke algebra.

In Section 7, we define rational Cherednik algebras for orbifolds. We also define the corresponding Hecke algebras, and define the KZ functor from the category of modules over the former to that over the latter. This generalizes to the "curved" case the KZ functor for rational Cherednik algebras of complex reflection groups, defined in Section 6. We then apply the KZ functor to showing that if the universal cover of the orbifold in question has a trivial H^2 (with complex coefficients), then the orbifold Hecke algebra is formally flat, and explain why the condition of trivial H^2 cannot be dropped. Next, we list examples of orbifold Hecke algebras which satisfy the condition of vanishing H^2 (and hence are formally flat). These include usual, affine, and double affine Hecke algebras, as well as Hecke algebras attached to Fuchsian groups, which include quantizations of del Pezzo surfaces and their Hilbert schemes; we work these examples out in some detail, highlighting connections with other subjects. Finally, we discuss the issue of algebraic flatness, and prove it in the case of algebras of rank 1 attached to Fuchsian groups, using the theory of deformations of group algebras of Coxeter groups developed in [ER].

In Section 8, we define symplectic reflection algebras (which inlude rational Cherednik algebras as a special case), and generalize to them some of the theory of Section 3. Namely, we use the theory of deformations of Koszul algebras to prove the PBW theorem for symplectic reflection algebras. We also determine the center of symplectic reflection algebras, showing that it is trivial when the parameter t is nonzero, and is isomorphic to the shperical subalgebra if $t = 0$. Next, we give a deformation-theoretic interpretation of symplectic reflection algebras as universal deformations of Weyl algebras smashed with finite groups. Finally, we discuss finite dimensional representations of symplectic reflection algebras for $t = 0$, showing that the Azumaya locus on the space of such representations coincides with the smooth locus. This uses the theory of Cohen-Macaulay modules and of homological dimension in commutative algebra. In particular, we show that for Cherednik algebras of type A_{n-1}, the whole representation space is smooth and coincides with the spectrum of the center.

In Section 9, we give another description of the spectrum of the center of the rational Cherednik algebra of type A_{n-1} (for $t = 0$), as a certain space of conjugacy classes of pairs of matrices, introduced by Kazhdan, Kostant, and Sternberg, and called the Calogero-Moser space (this space is obtained by classical hamiltonian reduction, and is a special case of a quiver variety). This yields a new construction of the Calogero-Moser integrable system. We also sketch a proof of the Gan-Ginzburg theorem claiming that the quotient of the commuting scheme by conjugation is reduced, and hence isomorphic to $\mathbb{C}^{2n}/\mathfrak{S}_n$. Finally, we explain that the Calogero-Moser space is a topologically trivial deformation of the Hilbert scheme of the plane, we use the theory of Cherednik algebras to compute the cohomology ring of this space.

In Section 10, we generalize the results of Section 9 to the quantum case. Namely, we prove the quantum analog of the Gan-Ginzburg theorem (the Harish Chandra-Levasseur-Stafford theorem), and explain how to quantize the Calogero-Moser space using quantum Hamiltonian reduction. Not surprisingly, this gives the same quantization as was constructed in the previous sections, namely, the spherical subalgebra of the rational Cherednik algebra.

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