THE KONTSEVICH MODULI SPACES OF STABLE MAPS

1. The Kontsevich moduli space of stable maps.

1.1. **Preliminaries.** We will begin with a detailed study of the Kontsevich moduli spaces of stable maps to $\mathbb{P}^r$. These spaces can be defined much more generally. However, we will have very little to say about the general situation. We will mostly concentrate on the case of genus zero maps to $\mathbb{P}^r$. The best introduction to Kontsevich moduli spaces is [FP] where you can find details about the construction of the space.

Definition 1.1. Let X be a smooth projective variety. Let $\beta \in H_2(X, \mathbb{Z})$ denote the class of a curve. The Kontsevich moduli space $\overline{M}_{g,n}(X, \beta)$ of n-pointed, genus g stable maps to X in the class parameterizes isomorphism classes of the following data

(1) $(C, p_1, \dots, p_n, f)$ an at worst nodal curve C of arithmetic genus g with n distinct, smooth points $p_1, \dots, p_n$ of C and a morphism $f : C \to X$ such that $f_*[C] = \beta$,
(2) The map is required to be stable; that is if f is constant on any component of C, then that component is required to have at least 3 distinguished points. The distinguished points are either marked points, or points lying over nodes in the normalization of the curve.

We have already encountered some examples of Kontsevich moduli spaces.

Example 1.2. The moduli space of stable maps to a point coincides with the moduli space of curves:
$$\overline{\mathrm{M}}_{g,n}(\mathbb{P}^0, 0) \cong \overline{\mathrm{M}}_{g,n}.$$

Example 1.3. The moduli space of degree zero stable maps, similarly, is easy to describe.
$$\overline{M}_{g,n}(X, 0) = \overline{M}_{g,n} \times X.$$
Since a degree 0 map from a connected curve is determined by specifying a point on X, this identification is immediate.

Example 1.4. The moduli space of degree one maps to $\mathbb{P}^r$ is isomorphic to the Grassmannian:
$$\overline{M}_{0,0}(\mathbb{P}^n, 1) = G(2, n+1) = \mathbb{G}(1, n).$$
A generalization of this example is the moduli space of degree one maps to a smooth quadric hypersurface Q in $\mathbb{P}^n$ for $n > 3$. In that case the Kontsevich moduli space is isomorphic to the orthogonal Grassmannian.

Example 1.5. The Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics or alternatively it is isomorphic to the blow up of the Hilbert scheme of conics in $\mathbb{P}^2$ along the Veronese surface of double lines.

Exercise 1.6. Prove the previous assertion by exhibiting a map (using the universal property of complete conics) from $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ to the space of complete conics. Check that this is a bijection on points. The claim then follows from Zariski's Main Theorem once we know that $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is smooth.

The main existence theorems for Kontsevich moduli spaces are the following. We refer you to [FP] for their proof.

Theorem 1.7. *If X is a complex, projective variety, then there exists a projective coarse moduli scheme $\overline{M}_{g,n}(X, \beta)$.*

Note that even when X is a nice, simple variety (such as $\mathbb{P}^2$), $\overline{\mathrm{M}}_{g,n}(X, \beta)$ may have many components of different dimensions.

Example 1.8. Consider the Kontsevich moduli space $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$ of genus one degree three stable maps to $\mathbb{P}^2$. This space has three components: two of dimension 9 and one of dimension 10. Naively, we might expect an open subset of $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$ to parameterize smooth cubic curves in $\mathbb{P}^2$. Indeed an open subset of one of the components does so. However, there is a second component whose general member is a map from a reducible curve with a genus zero component and a genus one component to $\mathbb{P}^2$ that contracts the genus one component and gives a degree three map on the genus zero component. Note that this component of $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$ has dimension 10. The dimension of rational cubics in $\mathbb{P}^2$ is 8, but the moduli of the contracted elliptic curve and the point of attachment add two more moduli. Similarly, one obtains a third component of dimension 9 by considering maps from elliptic curves with two rational tails which contract the elliptic curve and map the rational tails as a line and a conic.

Example 1.9. Even if we restrict ourselves to genus zero stable maps the Kontsevich moduli spaces may have many components of different dimensions. Consider degree two genus zero stable maps to a smooth degree seven hypersurface X in $\mathbb{P}^7$. Assume that X contains a $\mathbb{P}^3$. $\overline{\mathrm{M}}_{0,0}(X, 2)$ contains at least two components. One component covers X and has dimension 5. The conics in the $\mathbb{P}^3$ give a different component of dimension 8.

In order to obtain an irreducible moduli space with mild singularities one needs to impose some conditions on X. One possibility is to require that X is convex. Recall that a variety X is convex if for every map

$$f : \mathbb{P}^1 \to X,$$

f^*T_X is generated by global sections. Since every vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles, a variety is convex if for every map

$$f : \mathbb{P}^1 \to X,$$

the summands appearing in f^*T_X are non-negative. If we consider genus zero stable maps to convex varieties, the Kontsevich moduli space has very nice properties.

Theorem 1.10. *Let X be a smooth, projective, convex variety.*

(1) *$\overline{M}_{0,n}(X, \beta)$ is a normal, projective variety of pure dimension*

$$\dim(X) + c_1(X) \cdot \beta + n - 3.$$

(2) *$\overline{M}_{0,n}(X,\beta)$ is locally the quotient of a non-singular variety by a finite group. The locus of automorphism free maps is a fine moduli space with a universal family and it is smooth.*

(3) *The boundary is a normal crossings divisor.*

Observe that the previous theorem in particular applies to homogeneous varieties since homogeneous varieties are convex. In fact, if X is a homogeneous variety, then $\overline{\mathrm{M}}_{0,n}(X,\beta)$ is irreducible (see [KP]).

Remark 1.11. Although when we do not restrict ourselves to the case of genus zero maps to homogeneous varieties Kontsevich moduli spaces may be reducible with components of different dimensions, $\overline{\mathrm{M}}_{g,n}(X,\beta)$ possesses a virtual fundamental class of the expected dimension. The existence of the virtual fundamental class is the key to Gromov-Witten Theory.

Requiring a variety to be convex is a strong requirement on uniruled varieties. For instance, the blow-up of a convex variety ceasses to be convex. In fact, I do not know any examples of rationally connected, projective convex varieties that are not homogeneous.

Problem 1.12. Is every rationally connected, convex projective variety a homogeneous space? Either prove that it is or give counterexamples.

1.2. **Kontsevich's count of rational curves.** The Kontsevich moduli space is endowed with n evaluation morphisms

$$ev_i : \overline{M}_{g,n}(X,\beta) \to X,$$

where ev_i sends the point $(C, p_1, \ldots, p_n, f)$ to $f(p_i) \in X$.

From now on we will assume that X is a homogeneous variety and we will always restrict ourselves to the case of genus zero curves. Given the classes $\gamma_1, \ldots, \gamma_n$ of algebraic subvarieties of X, we can construct a class on $\overline{M}_{0,n}(X,\beta)$ by pulling them back via the evaluation morphisms and cupping:

$$ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n).$$

If the codimension of the classes add up to

$$\dim(X) + c_1(X) \cdot \beta + n - 3,$$

then we can define $I_\beta(\gamma_1, \ldots, \gamma_n)$, the Gromov-Witten invariant of X associated to the curve class β and cohomology classes $\gamma_1, \ldots, \gamma_n$ as follows:

$$I_\beta(\gamma_1, \ldots, \gamma_n) = \int_{\overline{\mathrm{M}}_{0,n}(X,\beta)} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n).$$

Remark 1.13. We can still define Gromov-Witten invariants for arbitrary, smooth projective varieties and higher genus curves. In that case we have to evaluate the product over the virtual fundamental class $[\overline{\mathrm{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}$ instead of $\overline{\mathrm{M}}_{0,n}(X,\beta)$.

The relation between Gromov-Witten invariants and enumerative geometry is established via the following variant of Kleiman's Transversality Theorem.

Lemma 1.14. *Let X be a homogeneous variety G/P. Let $\Gamma_1, \dots, \Gamma_n$ be irreducible subvarieties of X with classes $\gamma_1, \dots, \gamma_n$. Let $g_1 \cdots, g_n \in G$ be general elements, then the scheme theoretic intersection*

$$\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n) \tag{1}$$

is a finite number of reduced points supported in $M_{0,n}(X, \beta)$ and the Gromov-Witten invariant equals the cardinality of this set

$$I_\beta(\gamma_1, \cdots, \gamma_n) = \#\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n).$$

Example 1.15. In this example we derive Kontsevich's recursive formula for the number of rational plane curves of degree d that contain $3d-1$ general points. We begin by giving a geometric argument. We will then see how quantum cohomology gives the same answer formally. Define N_e to be the number of rational plane curves of degree e that contain $3e-1$ general points. Consider $3d$ pointed stable maps of degree d to $\mathbb{P}^2$ that map the points marked by $p_1, \dots, p_{3d-2}$ to fixed general points of $\mathbb{P}^2$. Fix also two general lines l_1, l_2 and require p_{3d-1} to map to l_1 and p_{3d} to map to l_2. Such stable maps give us a curve C in $\overline{\mathrm{M}}_{0,3d}(\mathbb{P}^2, d)$.

We will now analyze how C intersects the boundary divisors of $\overline{\mathrm{M}}_{0,3d}(\mathbb{P}^2, d)$. The main point is that there is a map

$$\pi : \overline{\mathrm{M}}_{0,n}(X, \beta) \to \overline{\mathrm{M}}_{0,4}$$

given by forgetting the map and the marked points but any specified four of the marked points (assuming of course that $n \geq 4$) and then stabilizing. Since the boundary divisors on $\overline{\mathrm{M}}_{0,4}$ are linearly equivalent, their pull-backs are also linearly equivalent.

Let us apply this discussion to our situation. Consider the map

$$\pi : \overline{\mathrm{M}}_{0,3d}(\mathbb{P}^2, d) \to \overline{\mathrm{M}}_{0,4}$$

as above that forgets all the points but p_1, p_2, p_{3d-1} and p_{3d}. The pull-back of the two divisors $\Delta_{\{p_1, p_{3d-1}\}, \{p_2, p_{3d}\}}$ and $\Delta_{\{p_1, p_2\}, \{p_{3d-1}, p_{3d}\}}$ are linearly equivalent, hence must intersect our curve C in the same number of points. Let us calculate these two numbers. First,

$$\pi^*\Delta_{\{p_1, p_{3d-1}\}, \{p_2, p_{3d}\}} = \sum_{\{i, A \ | \ \{p_1, p_{3d-1}\} \subset A, \{p_2, p_{3d}\} \subset A^c\}} \Delta_{i,A}$$

where the sum runs over boundary divisors in $\overline{\mathrm{M}}_{0,3d}(\mathbb{P}^2, d)$ consisting of maps with reducible domain curves such that the marking on one component contains p_1, p_{3d-1}, but does not contain p_2, p_{3d} and the map has degree $d-1 \geq i \geq 1$ on that component. The intersection of this divisor with our curve C is counted by the number of maps from reducible rational curves that have these properties.

Suppose the number of marked point on the component of degree i is larger than $3i$, then since more than $3i-1$ of these points are required to map to general fixed points of $\mathbb{P}^2$ by the above dimension count there will not be such maps. On the other hand, if there were fewer than $3i$ marked points, then the same argument when applied to the other component shows that there are no such maps. We conclude that $\#A = 3i$ and $\#A^c = 3(d-i)$. Since $\{p_1, p_{3d-1}\} \subset A, \{p_2, p_{3d}\} \subset A^c$ in order to determine the marking on the degree i component we need to choose $3i-2$ points among the $3d-4$ points $p_3, \dots, p_{3d-2}$. Once we choose those points, the number of rational plane curves passing through the $3i-1$ points is N_i. Each curve intersects l_1 in i points, hence the choice of point p_{3d-1} is i. Similarly the degree $d-i$ component contributes a factor of $N_{d-i}(d-i)$. Finally, in order to specify the map we have to specify among the $i(d-i)$ points of intersection between the two components which is the image of the node. We thus get that the total number of points of intersection of our curve C with this divisor is

$$\sum_{1\leq i\leq d-1} \binom{3d-4}{3i-2} i^2(d-i)^2 N_i N_{d-i}.$$

We now calculate the $C\cdot\pi^*\Delta_{\{p_1,p_2\},\{p_{3d-1},p_{3d}\}}$. We first observe that

$$\pi^*\Delta_{\{p_1,p_2\},\{p_{3d-1},p_{3d}\}} = \sum_{\{i,A \mid \{p_1,p_2\}\subset A^c, \{p_{3d-1},p_{3d}\}\subset A} \Delta_{i,A}$$

where the sum runs over $0\leq i\leq d-1$ and partitions of the marked points so that p_{3d-1}, p_{3d} are marked points in the domain on which the map has degree i and p_1, p_2 are not on that component. Note that since the images of p_1 and p_2 are distinct, $d-i$ cannot be zero. However, if the curve passes through the intersection point of l_1 and l_2, then the map may have a contracted component, where p_{3d-1} and p_{3d} lie on the component contracted to the point of intersection of l_1 and l_2. Hence, i may be zero. Keeping this in mind we see that the intersection of C with this divisor is

$$N_d + \sum_{1\leq i\leq d-1} \binom{3d-4}{3i-1} i^3(d-i) N_i N_{d-i}.$$

This is calculated in exactly the same way as above. Since these two divisors are linearly equivalent, the two numbers we calculated have to be equal. We conclude that the number of rational plane curves of degree d containing $3d-1$ general points may be recursively determined as follows:

$$N_d = \sum_{1\leq i\leq d-1} \left(\binom{3d-4}{3i-2} i^2(d-i)^2 - \binom{3d-4}{3i-1} i^3(d-i) \right) N_i N_{d-i}.$$

Of course, we know the first few of these numbers classically

$$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12.$$

Exercise 1.16. Check that N_2 and N_3 follow from the recursion and N_1. Calculate the next few N_d.

Exercise 1.17. Verify the details of the calculation above. In particular, carry out the necessary dimension counts that justify the claims made.

Exercise 1.18. Find the number of rational curves N_{d_1,d_2} in the class

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_1,d_2)$$

on $\mathbb{P}^1\times\mathbb{P}^1$ passing through $2d_1+2d_2-1$ general points using the same method.

Problem 1.19. Is it possible to generalize the previous discussion to other simple rational surfaces such as Hirzebruch surfaces or Del Pezzo surfaces? What kind of new problems arise? Are these surfaces convex?

1.3. **The quantum cohomology ring.** There is a way of formalizing the calculations we performed in the previous section. One forms a ring called the quantum cohomology ring whose structure constants encode the Gromov-Witten invariants. This ring turns out to be a commutative, associative ring with unit. The type of recursions we determined in the previous section then follows from the associativity relations in the ring.

We first choose a basis for the cohomology ring of the homogeneous variety X. We let $T_0 = 1, T_1, \ldots, T_m$ denote the divisor classes and $T_{m+1}, \ldots, T_r$ be an additive basis for the rest of the cohomology ring. There is a natural intersection matrix defined by

$$g_{ij} = \int_X T_i \cup T_j.$$

Let g^{ij} be the inverse of the intersection matrix. Then the products in the ordinary cohomology ring may be expressed as follows

$$T_i \cup T_j = \sum_{k,l} \left(\int_X T_i \cup T_j \cup T_k \right) g^{kl} T_l = \sum_{k,l} I_0(T_i, T_j, T_k) g^{kl} T_l.$$

The idea is to define a different multiplication structure on the cohomology ring by allowing Gromov-Witten invariants associated to non-zero curve classes as structure constants. Given a class γ in the cohomology ring define the generating function Φ by

$$\Phi(\gamma) = \sum_{n \geq 3} \sum_{\beta} \frac{1}{n!} \; I_\beta(\underbrace{\gamma, \ldots, \gamma}_{n \text{ times}}).$$

For convenience of notation $I_\beta(\underbrace{\gamma, \ldots, \gamma}_{n \text{ times}})$ is abbreviated by $I_\beta(\gamma^n)$, Setting

$$\gamma = \sum y_i T_i$$

and expanding, the function Φ becomes a formal power series in $\mathbb{Q}[[y_0, \ldots, y_r]]$

$$\Phi(y_0, \ldots, y_r) = \sum_{n_0 + \cdots + n_r \geq 3} \sum_{\beta} I_\beta(T_0^{n_0}, \ldots, T_r^{n_r}) \frac{y_0^{n_0} \cdots y_r^{n_r}}{n_0! \cdots n_r!}.$$

The third partial derivative of Φ with respect to y_i, y_j and y_k is

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k} = \sum_{n \geq 0} \sum_{\beta} \frac{1}{n!} I_\beta(\gamma^n, T_i, T_j, T_k).$$

Definition 1.20 (Quantum product)**.** Define a multiplication, called quantum multiplication, on $A^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[[y_0, \ldots, y_r]]$ by setting

$$T_i * T_j = \sum_{k,l} \Phi_{ijk} g^{kl} T_l$$

and extending the multiplication to $\mathbb{Q}[[y_0, \ldots, y_r]]$-linearly.

Theorem 1.21. *Under the quantum multiplication $A^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}[[y_0,\dots,y_r]]$ is a commutative, associative $\mathbb{Q}[[y_0,\dots,y_r]]$-algebra with unit T_0.*

Remark 1.22. The ring we have just defined is sometimes referred to as the big quantum cohomology ring. There is also a small quantum cohomology ring. The structure constants of the small quantum cohomology ring depend only on the three-pointed Gromov-Witten invariants. The definition of the small quantum multiplication differs from that of the big quantum multiplication only in the fact that in the definition of the small quantum cohomology we set the variables corresponding to classes of codimension two or more to zero. More precisely, set

$$\tilde{\Phi}_{ijk} = \Phi_{ijk}(y_0, y_1, \dots, y_m, 0, \dots, 0).$$

Define the small quantum product by

$$T_i * T_j = \sum_{k,l} \tilde{\Phi}_{ijk} g^{kl} T_l$$

Example 1.23 (Kontsevich's count revisited)**.** The quantum cohomology ring provides a formalism for deriving enumerative information about varieties. We demonstrate how this works in the case of $\mathbb{P}^2$. As a basis of the cohomology of $\mathbb{P}^2$ we can take $T_0 = 1$, $T_1 = [\text{line}]$, $T_2 = [\text{point}]$.

Note that if $\beta = 0$, the only way a Gromov-Witten invariant can be non-zero is if $n = 3$ and the codimension of the three cycles γ_1, γ_2 and γ_3 sum to the dimension of X. In this case, the Gromov-Witten invariant is the classical intersection

$$I_0(\gamma_1,\gamma_2,\gamma_3) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

Similarly, if one of the cohomology classes is the identity, the Gromov-Witten invariant vanishes unless $\beta = 0$, $n = 3$.

On the other hand, for $\mathbb{P}^2$ we have that $I_d(T_1^r, T_2^s) = 0$ unless $s = 2d-1$. If $s = 3d-1$, then

$$I_d(T_1^r, T_2^{3d-1}) = (rd)N_d.$$

Therefore, we obtain the following expression for the function Φ:

$$\begin{aligned}\Phi(y_0,y_1,y_2) &= \frac{y_0^2 y_2}{2} + \frac{y_0 y_1^2}{2} + \sum_{d\geq 1}\sum_{r\geq 0} I_d(T_1^r, T_2^{3d-1}) \frac{y_1^r}{r!} \frac{y_2^{3d-1}}{(3d-1)!} \\ &= \frac{y_0^2 y_2}{2} + \frac{y_0 y_1^2}{2} + \sum_{d\geq 1} N_d e^{dy_1} \frac{y_2^{3d-1}}{(3d-1)!}.\end{aligned}$$

We now express the quantum product of the generators.

$$T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_0.$$

Therefore, we have

$$\begin{aligned}(T_1 * T_1) * T_2 &= (T_2 + \Phi_{111} T_1 + \Phi_{112} T_0) * T_2 \\ &= \Phi_{221} T_1 + \Phi_{222} T_0 + \Phi_{111}(\Phi_{121} T_1 + \Phi_{122} T_0) + \Phi_{112} T_2\end{aligned}$$

On the other hand,

$$\begin{aligned} T_1 * (T_1 * T_2) &= T_1 * (\Phi_{121} T_1 + \Phi_{122} T_0) \\ &= \Phi_{121}(T_2 + \Phi_{111} T_1 + \Phi_{112} T_0) + \Phi_{122} T_1 \end{aligned}$$

By the associativity of the quantum cohomology ring the coeffiecents of T_i in the two expressions of $T_1 * T_1 * T_2$ have to be equal. Comparing the coefficients of T_0 (and remembering that taking the partial derivatives of Φ is independent of order), we obtain the relation

$$\Phi_{222} = (\Phi_{112})^2 - \Phi_{111}\Phi_{122}.$$

Working out these partial derivatives of Φ we obtain the equation

$$\begin{aligned} \textstyle\sum_{d\geq 1} N_d e^{dy_1} \frac{y_2^{3d-4}}{(3d-4)!} &= \\ \left(\textstyle\sum_{i\geq 1} N_i i^2 e^{iy_1} \frac{y_2^{3i-2}}{(3i-2)!}\right)^2 &- \left(\sum_{i\geq 1} N_i i^3 e^{iy_1} \frac{y_2^{3i-1}}{(3i-1)!}\right)\left(\sum_{i\geq 1} N_i i e^{iy_1} \frac{y_2^{3i-3}}{(3i-3)!}\right) \end{aligned}$$

Equating the coefficients it is easy to obtain Kontsevich's recursion

$$N_d = \sum_{1\leq i\leq d-1} \left(\binom{3d-4}{3i-2} i^2 (d-i)^2 - \binom{3d-4}{3i-1} i^3 (d-i) \right) N_i N_{d-i}.$$

Exercise 1.24. Work out recursion relations for the number of rational curves in the class $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_1, d_2)$ passing through $2d_1 + 2d_2 - 1$ general points in $\mathbb{P}^1 \times \mathbb{P}^1$ using the quantum cohomology formalism.

Exercise 1.25. Work out recursion relations for the number of rational curves of degree d in $\mathbb{P}^3$ that contain i general points and intersect $4d-2i$ general lines using the quantum cohomology formalism.

Exercise 1.26. Repeat the following two exercises for other simple varieties such as a smooth quadric threefold, the Grassmannian $G(2,4)$, ...

2. Divisor classes on the Kontsevich moduli space and enumerative geometry

In this section following Rahul Pandharipande [Pa] we determine the Picard group of the Kontsevich moduli space. We will then use this knowledge to study the enumerative geometry of rational curves in $\mathbb{P}^n$. In particular, we will solve some of the enumerative questions we asked earlier in the course about twisted cubics.

We start by giving the definitions of standard divisor classes.

(1) $\mathcal{H}$ is class of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$. This divisor is defined provided $r > 1$ and $d > 0$. Whenever we refer to $\mathcal{H}$ we assume these conditions hold.

(2) $\mathcal{L}_i = ev_i^*(\mathcal{O}_{\mathbb{P}^r}(1))$, for $1 \leq i \leq n$, are the n divisor classes obtained by pulling back $\mathcal{O}_{\mathbb{P}^r}(1)$ by the n evaluation morphisms.

(3) $\Delta_{(A,d_A),(B,d_B)}$ are the classes of boundary divisors consisting of maps with reducible domains. Here $A \sqcup B$ is any ordered partition of the marked points. d_A and d_B are non-negative integers satisfying $d = d_A + d_B$. If $d_A = 0$ (or $d_B = 0$), we require that $\#A \geq 2$ ($\#B \geq 2$, respectively).

Theorem 2.1 (Pandharipande)**.** *Let $r \geq 2$ and $d > 0$. The divisor class $\mathcal{H}$, the divisor classes $\mathcal{L}_i$ and the classes of boundary divisors $\Delta_{(A,d_A),(B,d_B)}$ generate the group of $\mathbb{Q}$-Cartier divisors of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$.*

Proof. We will prove a more precise version of the theorem and determine the relations between the divisors in the process. For simplicity let

$$P = \mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}.$$

Claim 2.2. *If the number of marked points $n \geq 3$, then $\mathcal{H}$ and the boundary divisors generate P.*

Consider the product of $n-3$ copies of $\mathbb{P}^1$. Let W be the complement of diagonals and the locus where one of the factors is $0, 1$ or ∞. Let U be the open subset

$$U \subset \mathbb{P} \oplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$$

parameterizing base-point free degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$. The complement of U has codimension at least 2. The product $W \times U$ embeds as an open subset of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ whose complement is the boundary. Since the group of codimension one cycles of $W \times U$ is generated by a multiple of $\mathcal{H}$, the claim follows.

Claim 2.3. *If the number of marked points $n = 2$, then the boundary, $\mathcal{L}_1$ and $\mathcal{L}_2$ generate P.*

Fix a hyperplane Λ. Consider the inverse image U of Λ under the third evaluation morphism from $\overline{\mathrm{M}}_{0,3}(\mathbb{P}^r, d)$. Away from the inverse image of the locus where the domain of the map is reducible and the images of the marked points lie in Λ, the forgetful map that forgets the third point is finite and projective. Hence it suffices to show that the divisor class group of this latter space is zero. This is clear.

Claim 2.4. *If the number of marked points $n = 1$, then the boundary, $\mathcal{L}_1$ and $\mathcal{H}$ generate P.*

In order to see this claim fix two general hyperplanes Λ_1, Λ_2 and carry out an argument similar to the previous two arguments.

Claim 2.5. *If the number of marked points $n = 0$, then $\mathcal{H}$ and the boundary divisors generate P.*

Fix three hyperplanes H_1, H_2, H_3. Consider the complement Z in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ of the boundary and the three hypersurfaces of maps intersecting $H_i \cap H_j$, $i \neq j$. It suffices to prove that the divisor classes of Z is trivial. This is easy to see.

Note that the previous four claims suffice to complete the proof of the theorem. □

These divisors satisfy certain relations. Already this is clear from the proof of the theorem. These relations may be determined as follows.

Relations among the boundary divisors. The Kontsevich moduli space admits a morphism to the Deligne-Mumford moduli space of stable curves $\overline{\mathrm{M}}_{0,n}$ by forgetting the map and stabilizing. We already know relations among the boundary components of $\overline{\mathrm{M}}_{0,n}$. Pulling back these relations among the boundary components yields the relations among the boundary components.

Exercise 2.6. By exhibiting one parameter families that have different intersection numbers show that

(1) $\mathcal{H}$ is not in the span of boundary divisors. (Hint: Consider the Veronese image of a pencil of lines in $\mathbb{P}^2$)

(2) If the number of marked points is one, then $\mathcal{H}$ and $\mathcal{L}_1$ are independent modulo the boundary.

(3) If the number of marked points is two, then $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent modulo the boundary.

Exercise 2.7. Fix a hyperplane Λ in $\mathbb{P}^r$. Show that the locus of stable maps in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ where $f^{-1}(\Lambda)$ is not d distinct, smooth points is a divisor $\mathcal{T}$ in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$. Calculate the class of this divisor in terms of $\mathcal{H}$ and the boundary divisors. (Hint:

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{i=1}^{\lfloor d/2 \rfloor} \frac{i(d-i)}{d}\Delta_i.)$$

2.1. **An algorithm for computing the genus zero characteristic numbers in projective space.** There is an algorithm for computing the number of rational curves in $\mathbb{P}^r$ that intersect i general codimension two linear spaces and are tangent to $(r+1)(d+1)-4-i$ general hyperplanes. In general this algorithm gets out of hand very quickly and it is hard to implement. However, for small degree curves it solves the characteristic number problem rather easily.

Proposition 2.8. *The number of rational curves of degree d in $\mathbb{P}^r$ that intersect i general codimension two linear spaces and are tangent to $(r+1)(d+1)-4-i$ general hyperplanes may be computed as $\mathcal{H}^i \cdot \mathcal{T}^{(r+1)(d+1)-4-i}$ on $\overline{M}_{0,0}(\mathbb{P}^r, d)$.*

Assuming the proposition for the moment, we can describe the algorithm. We can compute the intersections of top monomials consisting of $\mathcal{H}$ and $\mathcal{L}_i$. (For instance we can use the associativity relations in the cohomology ring and Kontsevich-Manin's First Reconstruction Theorem in order to determine these top degree monomials.)

In order to determine the top monomials involving the boundary, we can pull-back to the boundary divisors. The boundary itself is a product of Kontsevich moduli spaces. We can express the pull-back of the standard divisors as standard divisors on the product and proceed inductively.

Exercise 2.9. Determine the characteristic numbers of conics in $\mathbb{P}^2$ using this algorithm. In particular, show that

$$\mathcal{H}^5 = \mathcal{T}^5 = 1, \quad \mathcal{H}^4\mathcal{T} = \mathcal{H}\mathcal{T}^4 = 2, \quad \mathcal{H}^3\mathcal{T}^2 = \mathcal{H}^2\mathcal{T}^3 = 4.$$

Exercise 2.10. Show that the class of degree two maps whose image is tangent to a conic has class

$$2(\mathcal{H} + \mathcal{T})$$

in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$. Using this fact and the previous exercise, show that there are 3264 conics tangent to 5 general conics in $\mathbb{P}^2$.

Exercise 2.11. Determine the number of twisted cubics in $\mathbb{P}^3$. intersecting i general lines and tangent to $12 - i$ general planes by applying the algorithm described in this section. (Hint: The numbers are determined by $\mathcal{H}^i\mathcal{T}^{12-i}$. In order of decreasing i they are 80160, 134400, 209760, 297280, 375296, 415360, 401920, 343360, 264320, 188256, 128160, 85440 and 56960.)

Exercise 2.12. Show that the closure of the locus of twisted cubics tangent to a smooth quadric hypersurface is a divisor with class $2\mathcal{H} + 2\mathcal{T}$. Using the previous exercise determine the number of twisted cubics tangent to 12 general quadric hypersurfaces. (Hint: The number is equal to $(2\mathcal{H} + 2\mathcal{T})^{12}$. You should get 5,819,539,783,680.)

Exercise 2.13. Finally, establish the proposition that guarantees that the characteristic numbers are indeed given by the claimed intersection numbers. First, show that the divisors $\mathcal{H}, \mathcal{T}$ and $\mathcal{L}_i$ are base-point-free divisors. Conclude from this that if representatives defined with respect to general linear spaces are chosen, then the intersections are zero dimensional. Furthermore, check that the points of intersection correspond to maps that in addition have irreducible domain, are simply tangent to those hyperplanes defining $\mathcal{T}$ and intersect the linear spaces defining $\mathcal{H}$ transversely. Finally apply Kleiman's Bertini Theorem to the universal map in order to deduce that the points occuring in the intersection are reduced.

3. Counting genus zero curves in $\mathbb{P}^n$: Vakil's algorithm

Ravi Vakil in his thesis developed a different approach for calculating genus zero Gromov-Witten invariants using degenerations. Following [V] we describe his method. For proofs and further discussions we refer you to Ravi's paper.

Before we describe his theorem that allows us to do the following computations, we will give a few sample calculations to indicate how his method works.

Example 3.1. Let us find out the number of conics in $\mathbb{P}^3$ that contain 2 general points p_1, p_2 and intersect 4 general lines $l_1, \ldots, l_4$. The idea is to specialize the conditions that the curves satisfy one at a time to general linear spaces of a hyperplane. Fix a general hyperplane H, that is a general $\mathbb{P}^2$. We can assume that H contains the two points p_1, p_2. We specialize one of the lines l_1 to H. Any connected degree two curve containing p_1, p_2 and intersecting l_1 either has to be contained in H or it has to have a component in H. In the first case the conic is uniquely determined.

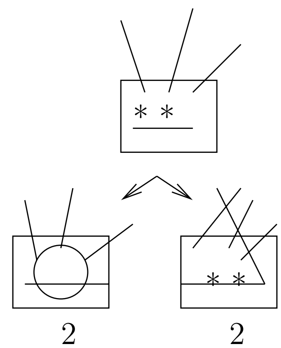

FIGURE 1. Calculating the number of conics that contain 2 general points and intersect 4 general lines.

It has to contain the two points in H and the three points of intersection of l_2, l_3, l_4 with H. We count this conic twice for the choice of intersection of the conic with l_1. In the latter case the component in H has to be a line. In fact, it has to be the line passing through p_1 and p_2. The remaining component has to intersect this line and l_2, l_3, l_4. There are two lines in $\mathbb{P}^3$ that intersect 4 lines. We conclude that there are a total of 4 conics that contain 2 general points and intersect 4 general lines. See Figure 1 for a schematic representation of this calculation.

Example 3.2. Let us calculate that there are 5 twisted cubics in $\mathbb{P}^3$ containing 5 general points and intersecting 2 general lines. We will carry out this calculation in two different ways in order to show the degenerations that can occur. Figure 2 shows a schematic diagram of both of these degenerations.

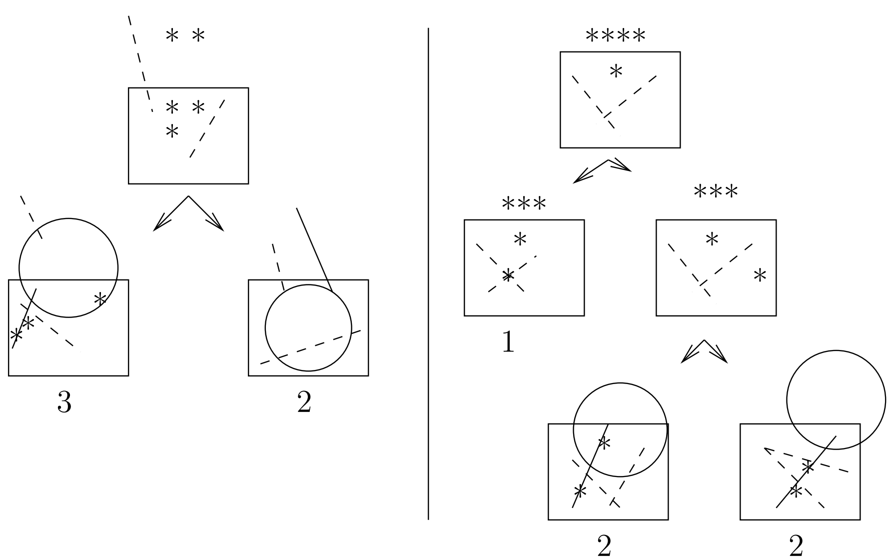

FIGURE 2. Calculating the number of twisted cubics that contain 5 general points and intersect 2 general lines.

The left hand panel shows the degeneration when we first specialize the line l_1 to a plane P that is spanned by the three points p_1, p_2, p_3. Once we make this degeneration, the limit twisted cubics necessarily become reducible. P contains either a conic or a line. If P contains a conic, then the residual line has to be the span of the remaining two points p_4, p_5 not contained in P. The conic then is uniquely determined by the facts that it has to intersect this line, l_2 and contain p_1, p_2, p_3. This solution contributes 2 for the choice of intersection of the conic with l_1. If P contains a line, the line has to be the span of two of the points p_1, p_2, p_3, hence there are 3 choices for the line. The conic is then uniquely determined by the requirements that it intersect the line in P, l_2 and contain the remaining three points. We see that there are 5 twisted cubics that contain 5 general points and intersect 2 general lines.

The right hand panel shows a different order of degeneration for the same problem. We first specialize a point and the two lines to a general plane P. The limiting twisted cubics may meet l_1 and l_2 along their points of intersection. This problem reduces to counting twisted cubics passing through 6 general points. The answer is 1. Otherwise, we specialize another point to P. At this stage the limiting twisted cubics have to become reducible. There could be a line in P (necessarily the span of the two points contained in P) and a conic in the plane spanned by the three points not contained in P. A priori there seems to be a one parameter family of possible conics.

This forces us to answer the question of which among these conics are limits of our original solutions. The key to the answer lies in tracing the limit of the Cartier divisor cut out on the family of twisted cubics by the plane P. The limit is a degree three divisor on the limiting curve. However, the restriction of the limiting divisor to the reducible curve may have degree 2 or 3 on the line component. If it has degree 2, then the conic has to intersect one of l_1 or l_2 giving two solutions. If it has degree 3, the conic has to be tangent to the plane P. There is one such conic. However, in this case there is a new twist. Two distinct solutions approach this solution. Hence, this solution counts with multiplicity 2.

For more examples see [V]. We now describe how the algorithm in the previous examples works in general. The aim is to calculate the characteristic numbers of rational curves in projective space. Recall that the characteristic numbers of rational curves of degree e are the numbers of rational curves of degree e that intersect general linear subspaces Λ_i of $\mathbb{P}^n$ of codimension c_i such that

$$\sum_i (c_i - 1) = (e+1)(n+1) - 4.$$

In fact, the algorithm will calculate slightly more general numbers by allowing the curves to have higher order contact with a fixed hyperplane.

The idea is to specialize the linear spaces that impose conditions on the curves one at a time to general linear spaces of a fixed hyperplane H. We then trace the limits of the stable maps.

More precisely, fix positive integers d and r. Let $\{\Delta_i\}_{i \in I}$ be a general collection of linear subspaces of $\mathbb{P}^r$ and let $\{\Gamma_j^m\}_{j \in J}$ be a general collection of linear subspaces of a hyperplane H in $\mathbb{P}^r$. Let $X_r(d, \Gamma, \Delta)$ be the locus of stable maps of degree d to $\mathbb{P}^r$ with $\#I + \#J$ marked points such that the point p_i maps to Δ_i and the point q_j^m

maps to Γ_j^m. Furthermore, assume that the pull-back of the hyperplane H under the stable map is $\sum_j mq_j^m$. You should think of $X_r(d, \Gamma, \Delta)$ as parameterizing rational curves of degree d with specified contact orders with a hyperplane H along general linear subspaces Γ_j^m of H and intersting other linear subspaces Δ_i of $\mathbb{P}^r$. There is a Cartier divisor D_H of $X_r(d, \Gamma, \Delta)$ obtained by requiring one of the points not mapping to H to map to H.

The task at hand is to enumerate the Weil divisors (together with their multiplicities) that form the components of D_H. The following loci turn out to be crucial. Let

$$Y_r(d(0), \Gamma(0), \Delta(0); \ldots; d(l), \Gamma(l), \Delta(l))$$

be the locus of stable maps to $\mathbb{P}^r$ such that

(1) The domain has $l+1$ components. The central component is $C(0)$ and all other components meet this component.

(2) The map has degree $d(i)$ on the ith component.

(3) There is a partition of the conditions Δ and Γ to the various components and the images of the marked points on the component $C(i)$ lie in the corresponding linear constraints $\Delta(i)$ and $\Gamma(i)$.

(4) The only component that is mapped to H is $C(0)$. All the other components intersect H along the marked points and the point of attachment of $C(i)$ with $C(0)$.

(5) The pull-back of H to the ith component by the stable map has the form

$$\sum mp_j^m(i) + m_i(C(0) \cap C(i))$$

where the positive integer m_i is defined by

$$m_i = d(i) - \sum_m m\#\Gamma_i^m.$$

The following theorem of Vakil identifies the components of D_H.

Theorem 3.3 (Vakil)**.** *Every component of D_H has the form*

$$Y_r(d(0), \Gamma(0), \Delta(0); \ldots; d(l), \Gamma(l), \Delta(l))$$

for some partition of d into non-negative integers and partitions of Δ and Γ. The component

$$Y_r(d(0), \Gamma(0), \Delta(0); \ldots; d(l), \Gamma(l), \Delta(l))$$

occurs with multiplicity $\prod m_i$.

We can depict Vakil's theorem rather informally by the diagram in Figure 3:

Every limiting curve that occurs has the form that there is one central component contained in the hyperplane and some number (r in the picture) of irreducible components that are not contained in the hyperplane and intersect the central component. In addition each of these latter components have contact of order m_i with the hyperplane. Vakil's theorem says that such a limit occurs with multiplicity $\prod m_i$.

The proof is non-trivial. The task is to express the Cartier divisor D_H as a There is an easy component of the proof. One identifies the potential limits by a dimension count. The limit has to be a stable map from a tree of $\mathbb{P}^1$s. Once we fix

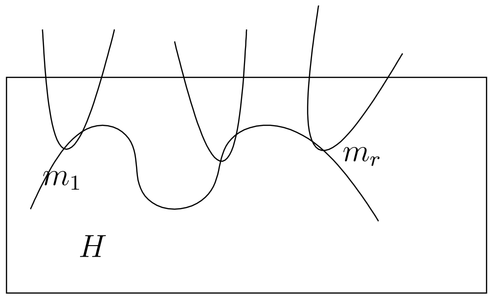

FIGURE 3. The limits that occur.

the combinatorics of the tree the dimension of such maps is easy to calculate. The problem is that contracted components may add moduli. However, these loci of maps are not enumeratively relevant because their image in the Hilbert scheme or the Chow variety have smaller dimension. Keeping this in mind it is easy to see that the loci described in the theorem are the only enumeratively relevant codimension one loci that can occur in the expression of D_H.

The technically harder part of the proof is to calculate the multiplicity of each of the enumeratively relevant Weil divisors. One first reduces the problem when the target is $\mathbb{P}^1$ instead of $\mathbb{P}^r$. This is done by projecting via a general codimension two linear space contained in the hyperplane H. This is only a rational map, but the locus where it is defined intersects all the enumeratively relevant divisors and is smooth at a general point of the intersection. The problem thus reduces to analyzing coverings of $\mathbb{P}^1$. In this setting the calculation of the deformation spaces is easier and yields the desired multiplicity.

Exercise 3.4. Determine the number of conics in $\mathbb{P}^3$ intersecting i general points and $8-2i$ general lines for $0 \geq i \geq 3$ using Vakil's method. Try this with different orders of degeneration. Which ones tend to be easier to carry out? Compare your results with those obtained by calculating in the cohomology ring of the Hilbert scheme of conics in $\mathbb{P}^3$.

Exercise 3.5. Using Vakil's method show that there is a unique twisted cubic containing 6 general points in $\mathbb{P}^3$. By induction show that there is a unique rational normal curve of degree d in $\mathbb{P}^d$ containing $d+3$ points. Give a direct argument that does not use degenerations.

Exercise 3.6. Show that there are 5 twisted cubics that contain 5 points and meet two lines; and 30 twisted cubics that contain 4 points and meet 4 lines. Try doing these calculations with different orders of degeneration. Using induction deduce that the number of rational normal curves of degree d in $\mathbb{P}^d$ that contain $d+2$ general points, meet a line and a $\mathbb{P}^{d-2}$ is $(d^2+d-2)/2$.

Exercise 3.7. Degeneration techniques may be used to calculate tangency to higher degree hypersurfaces as well. Show that there are 3264 conics in $\mathbb{P}^2$ tangent to five general conics. Do this by degenerating the conics into a pair of lines.

(Hint: In the limit the conics maybe tangent to either of the two lines or pass through the singular point. The latter count with multiplicity 2.) Do this calculation directly in the cohomology ring of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$, recalling that the Kontsevich space in this case is isomorphic to the blow up of $\mathbb{P}^5$ (the Hilbert scheme) along the Veronese surface of double lines.

Remark 3.8. R. Vakil using the same technique can also calculate the characteristic numbers of elliptic curves. The details are very similar. A few new phenomena (such as the need to record some information about the Picard group of the elliptic curve) complicate matters slightly. We leave it to you to develop or read the necessary modifications.

Exercise 3.9. Try finding the number of elliptic cubic curves in $\mathbb{P}^3$ that contain 2 general points and intersect 8 general lines. (Hint: Specialize the lines one at a time to a plane P containing the two points. If after specializing l_1 to P the elliptic cubics do not have a component in P where do they have to intersect l_1?)

Problem 3.10. Extend Vakil's method to higher genus curves. It would be especially interesting to be able to determine the characteristic numbers of canonical curves or curves embedded by special g^r_d's using degenerations. At present this problem seems difficult.

Remark 3.11. Caporaso and Harris in [CH1] (see also [CH2]) using essentially the same technique (but working in a partial compactification of the Severi variety rather than the Kontsevich moduli space) calculated the degrees of Severi varieties in $\mathbb{P}^2$ for all genera. I believe this work inspired Vakil to develop his algorithm.

Exercise 3.12. Using degenerations show that there are 2^6 canonical curves of genus 4 in $\mathbb{P}^3$ containing 9 general points and meeting 6 lines. Determine the number of canonical curves of genus 4 in $\mathbb{P}^3$ contining 8 general points and intersecting 8 general lines. (Hint: Use the fact that a genus 4 curve is the complete intersection of a quadric and a cubic surface and trace the limit of the quadric surface.)

Remark 3.13. Degeneration techniques may be used much more generally to determine the characteristic numbers of varieties. We already used this technique to obtain Littlewood - Richardson rules for Grassmannians. It is possible to calculate certain characteristic numbers of scrolls and other simple surfaces such as Del Pezzo surfaces.

4. The cones of ample and effective divisors on the Kontsevich moduli space

In this section we will discuss the ample cone and the effective cone of divisors on the Kontsevich moduli space of genus zero stable maps to $\mathbb{P}^r$. For more details you can consult [CHS1] and [CHS2].

4.1. **The ample cone of the Kontsevich moduli space.** We begin by describing the ample cone of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$.

Theorem 4.1. *Let r and d be positive integers, n a nonnegative integer such that $n + d \geq 3$. There is an injective linear map,*

$$v : Pic(\overline{\mathcal{M}}_{0,n+d})_{\mathbb{Q}}^{\mathfrak{S}_d} \to Pic(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))_{\mathbb{Q}}.$$

The NEF cone of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is the product of the cone generated by

$$\mathcal{H}, \mathcal{T}, \mathcal{L}_1, \ldots, \mathcal{L}_n$$

and the image under v of the NEF cone of $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$.

We recall that $\mathcal{H}$ is the class of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$ (provided $r > 1$ and $d > 0$). The class $\mathcal{L}_i$ is the pullback of $\mathcal{O}_{\mathbb{P}^r}(1)$ by the i^{th} evaluation morphism. Fixing a hyperplane $\Pi \subset \mathbb{P}^r$, $\mathcal{T}$ is the class of the divisor parametrizing stable maps $(C, p_1, \ldots, p_i, f)$ for which $f^{-1}(\Pi)$ is not simply d reduced, smooth points of C. In terms of Pandharipande's generators, the class of $\mathcal{T}$ equals,

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{k(d-k)}{d} (\sum_{A,B} \Delta_{(A,k),(B,d-k)}).$$

We now describe the map v that occurs in Theorem 4.1.

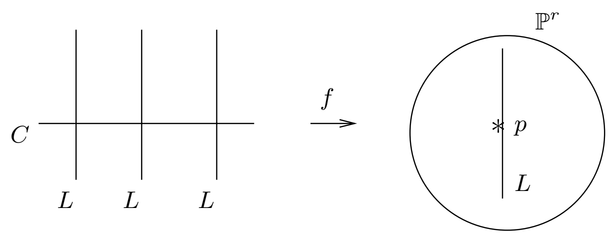

FIGURE 4. The morphism α.

The morphism α. There is a 1-morphism $\alpha : \overline{M}_{0,n+d} \times \mathbb{P}^{r-1} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Fix a point $p \in \mathbb{P}^r$ and a line $L \subset \mathbb{P}^r$ containing p. To every curve C in $\overline{M}_{0,n+d}$ attach a copy of L at each of the last d marked points and denote the resulting curve by C'. Consider the morphism $f : C' \to \mathbb{P}^r$ that contracts C to p and maps the d rational tails isomorphically to L (see Figure 4). Since the space of lines in $\mathbb{P}^r$ passing through the point p is parameterized by $\mathbb{P}^{r-1}$, there is an induced 1-morphism $\alpha : \overline{M}_{0,n+d} \times \mathbb{P}^{r-1} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

Since α is invariant for the action of $\mathfrak{S}_d$ permuting the last d marked points, the pull-back map determines a homomorphism

$$\alpha^* = (\alpha_1^*, \alpha_2^*) : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}).$$

We will denote the two projections of α^* by α_1^* and α_2^*.

The morphisms β_i. For each $1 \leq i \leq n$, there is a 1-morphism $\beta_i : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Fix a degree-$(d-1)$, $(n-1)$-pointed curve C containing all except the i^{th} marked point. At a general point of C, attach a line L. Attach a line L to C at a general point of C. The resulting degree-d, reducible

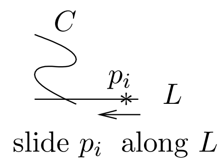

FIGURE 5. The morphism β_i.

curve will be the domain of our map. The final, i^{th} marked point is in L. Varying p_i in L gives a 1-morphism $\beta_i : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ (see Figure 5). This definition has to be slightly modified in the cases $(n, d) = (1, 1)$ or $(2, 1)$. When $(n, d) = (1, 1)$, we assume that the line L with the varying marked point p_i constitutes the entire stable map. When $(n, d) = (2, 1)$, we assume that the map has L as the only component. One marked point is allowed to vary on L and the remaining marked point is held fixed at a point $p \in L$.

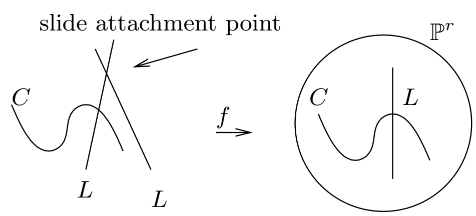

FIGURE 6. The morphism γ.

The morphism γ. If $d \geq 2$, there is a 1-morphism $\gamma : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Take two copies of a fixed line L attached to each other at a variable point. Fix a point p in the second copy of L. Let C be a smooth, degree-$(d-2)$, genus 0, $(n+1)$-pointed stable map to $\mathbb{P}^r$ whose $(n+1)$-st point maps to p. Attach this to the second copy of L at p. Altogether, this gives a degree-d, n-pointed, genus 0 stable maps with three irreducible components. The n marked points are the first n marked points of C. The only varying aspect of this family of stable maps is the attachment point of the two copies of L. Varying the attachment point in $L \cong \mathbb{P}^1$ gives a stable maps is parameterized by $\mathbb{P}^1$, hence there is an induced 1-morphism $\gamma : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ (see Figure 6). When $(n, d) = (1, 2)$, we modify the definition by assuming that the map consists only of the two copies of the line L and the marked point is held fixed at the point p on the second copy of L.

If $d \geq 2$, denote by $P_{r,n,d}$ the Abelian group

$$P_{r,n,d} := \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}) \times \mathrm{Pic}(\mathbb{P}^1)^n \times \mathrm{Pic}(\mathbb{P}^1).$$

Denote by $u = u_{r,n,d} : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \to P_{r,n,d}$ the pull-back map

$$u_{r,n,d} = (\alpha^*, (\beta_1^*, \ldots, \beta_n^*), \gamma^*).$$

If $d = 1$, denote by $P_{r,n,1}$ the Abelian group

$$P_{r,n,1} := \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}) \times \mathrm{Pic}(\mathbb{P}^1)^n$$

Divisors in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$	α_1^*	α_2^*	β_i^*	γ^*
$\mathcal{T}$	0	0	0	$\mathcal{O}_{\mathbb{P}^1}(2)$
$\mathcal{H}$	0	$\mathcal{O}_{\mathbb{P}^{r-1}}(d)$	0	0
$\mathcal{L}_i$	0	0	$\mathcal{O}_{\mathbb{P}^1}(1)$	0
$\mathcal{L}_{j\neq i}$	0	0	0	0
$\Delta_{(\emptyset,1),(\underline{n},d-1)}$	c	$\mathcal{O}_{\mathbb{P}^{r-1}}(-d)$	$\mathcal{O}_{\mathbb{P}^1}(-1)$	$\mathcal{O}_{\mathbb{P}^1}(4)$
$\Delta_{(\emptyset,2),(\underline{n},d-2)}$	$\tilde{\Delta}_{(\emptyset,2),(\underline{n},d-2)}$	0	0	$\mathcal{O}_{\mathbb{P}^1}(-1)$
$\Delta_{(\{i\},1),(\{i\}^c,d-1)}$	$\tilde{\Delta}_{(\{i\},1),(\{i\}^c,d-1)}$	0	$\mathcal{O}_{\mathbb{P}^1}(-1)$	0
$\Delta_{(A,d_A),(B,d_B)}$ all others	$\tilde{\Delta}_{(A,d_A),(B,d_B)}$	0	0	0

FIGURE 7. The pull-backs of the standard generators

and denote by $u = u_{r,n,1} : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)) \to P_{r,n,1}$ the pull-back map

$$u_{r,n,1} = (\alpha^*, (\beta_1,^*, \dots, \beta_n^*))$$

Theorem 4.1 is equivalent to the following.

Theorem 4.2. *The map $u_{r,n,d} \otimes \mathbb{Q} : Pic(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))_{\mathbb{Q}} \to P_{r,n,d} \otimes \mathbb{Q}$ is an isomorphism. The image under $u_{r,n,d} \otimes \mathbb{Q}$ of the ample cone, resp. NEF, eventually free cone of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ equals the product of the ample cones, resp. NEF, eventually free cones of $Pic(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $Pic(\mathbb{P}^{r-1})$, and the factors $Pic(\mathbb{P}^1)$.*

To apply Theorem 4.2, we need to express the images of the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ in terms of the standard generators for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $\mathrm{Pic}(\mathbb{P}^{r-1})$ and $\mathrm{Pic}(\mathbb{P}^1)$ factors. This is summarized in Table 7.

Let $\Pi \subset \mathbb{P}^r$ be a hyperplane not containing the point p used to define the morphisms α and γ. Assume that the degree $d-1$ curve used to define the morphisms β_i is not tangent to Π, and none of the marked points on this curve are contained in Π. Finally, assume that the degree $d-2$ curve used to define the morphism γ is not tangent to Π and none of the marked points are contained in Π.

Denote by $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)$ the open substack of $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)$ parameterizing stable maps with irreducible domain. Let

$$\mathrm{ev}_{n+1,\dots,n+d} : \mathcal{M}_{0,n+d}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^d$$

be the evaluation morphism associated to the last d marked point. Denote by $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ the inverse image of Π^d and by $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ the closure of $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ in $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)$.

$\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d)_\Pi$ is $\mathfrak{S}_d$-invariant under the action of $\mathfrak{S}_d$ on $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d)$ permuting the last d marked points. Denote by

$$\pi : \overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$$

the forgetful 1-morphism that forgets the last d marked points and stabilizes the resulting family of prestable maps. This is $\mathfrak{S}_d$-invariant. Denote by

$$\rho : \overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d) \to \overline{\mathrm{M}}_{0,n+d}$$

the 1-morphism that stabilizes the universal family of marked prestable curves over $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d)$. This is $\mathfrak{S}_d$-equivariant.

Denote by $q : \overline{\mathrm{M}}_{0,n+d} \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ the geometric quotient. The composition $q \circ \rho : \overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d)_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ is $\mathfrak{S}_d$-equivariant. Because $\mathcal{M}_{0,n+d}(\mathbb{P}^r,d)_\Pi$ is an $\mathfrak{S}_d$-torsor over O_Π, there is a unique 1-morphism $\phi'_\Pi : O_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ such that $\phi' \circ \pi = q \circ \rho$.

Definition 4.3. Define U_Π to be the maximal open substack of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ over which ϕ'_Π extends to a 1-morphism, denoted

$$\phi_\Pi : U_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d.$$

Define I_Π to be the normalization of the closure in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \times \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ of the image of the graph of ϕ'_Π, i.e., I_Π is the normalization of the image of $(\pi, q\circ\rho)$. Define $\widetilde{I}_\Pi$ to be the normalization of the image of (π,ρ) in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \times \overline{\mathrm{M}}_{0,n+d}$. Finally, define $\widetilde{U}_\Pi$ to be the inverse image of U_Π in $\widetilde{I}_\Pi$.

There is a pull-back map of $\mathfrak{S}_d$-invariant invertible sheaves,

$$\rho^* : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\widetilde{I}_\Pi)^{\mathfrak{S}_d},$$

which further restricts to $\mathrm{Pic}(\widetilde{U}_\Pi)^{\mathfrak{S}_d}$. After étale base-change from U_Π to a scheme, the morphism $\widetilde{U}_\Pi \to U_\Pi$ is the geometric quotient of $\widetilde{U}_\Pi$ by the action of $\mathfrak{S}_d$. Therefore the pull-back map $\mathrm{Pic}(U_\Pi) \to \mathrm{Pic}(\widetilde{U}_\Pi)^{\mathfrak{S}_d}$ is an isomorphism after tensoring with $\mathbb{Q}$; in fact, both the kernel and cokernel are annihilated by $d!$. Because $\overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ is a proper scheme and because $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is separated and normal, by the valuative criterion of properness the complement of U_Π has codimension ≥ 2. The smoothness of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ and [Ha, Prop. 6.5(c)] imply that the restriction map $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \to \mathrm{Pic}(U_\Pi)$ is an isomorphism.

Definition 4.4. Define $v : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}$ to be the unique homomorphism commuting with ρ^* via the isomorphisms above.

The map v is independent of the choice of Π, hence it sends NEF divisors to NEF divisors.

Lemma 4.5. *For every base-point-free invertible sheaf $\mathcal{L}$ in $Pic(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $v(\mathcal{L})$ is base-point-free. In particular, for every ample invertible sheaf $\mathcal{L}$, $v(\mathcal{L})$ is NEF. Thus, by Kleiman's criterion, for every NEF invertible sheaf $\mathcal{L}$, $v(\mathcal{L})$ is NEF.*

Proof. For every $[(C,(p_1,\dots,p_n),f)]$ in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$, there exists a hyperplane Π satisfying the conditions above and such that $f^{-1}(\Pi)$ is a reduced Cartier divisor containing none of $p_1,\dots,p_n$. $(C,(p_1,\dots,p_n),f)$ is contained in U_Π. Since $\mathcal{L}$ is base-point-free, there exists a divisor D in the linear system $|\mathcal{L}|$ not containing $\phi_\Pi[(C,(p_1,\dots,p_n),f)]$. By the proof of [Ha, Prop. 6.5(c)], the closure of $\phi_\Pi^{-1}(D)$ is in the linear system $|v(\mathcal{L})|$; and it does not contain $[(C,(p_1,\dots,p_n),f)]$. □

Lemma 4.6. (i) *The images of α, β_i and γ are contained in U_Π.*

(ii) *The morphisms $\phi_\Pi \circ \beta_i$ and $\phi_\Pi \circ \gamma$ are constant morphisms. Therefore $\beta_i^* \circ v$ and $\gamma^* \circ v$ are the zero homomorphism.*

(iii) *The composition of α with ϕ_Π equals $q \circ pr_{\overline{M}_{0,n+d}}$. Therefore*

$$\alpha^* \circ v : Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \to Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \times Pic(\mathbb{P}^{r-1})$$

is the homomorphism whose projection on the first factor is the identity, and whose projection on the second factor is 0.

Proof. **(i):** The image of α is contained in O_Π. Denote by q the intersection point of L and Π.

The image $\beta_i(L - \{q\})$ is contained in O_Π. The stable map $\beta_i(q)$ sends the i^{th} marked point into Π. Up to labeling the d points of the inverse image of Π, there is only one $(n+d)$-pointed stable map in $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ that stabilizes to this stable map. It is obtained from $\beta_i(q)$ by removing the i^{th} marked point from L, attaching a contracted component C' to L at q, containing the i^{th} marked point and exactly one of the last d marked points, and labeling the $d-1$ points in $C \cap \Pi$ with the remaining $d-1$ marked points.

Similarly, $\gamma(L - \{q\})$ is contained in O_Π. The stable map $\gamma(q)$ has two copies of L attached to each other at q. This appears to be a problem, because the inverse image of $\gamma(q)$ in $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ is 1-dimensional, isomorphic to $\overline{\mathrm{M}}_{0,4}$. The stable maps have a contracted component C' such that both copies of L are attached to C' and 2 of the d new marked points are attached to C'. The remaining $d-2$ marked points are the points of $C \cap \Pi$. However, the map ρ that stabilizes the resulting prestable $(n+d)$-marked curve is constant on this $\overline{\mathrm{M}}_{0,4}$. Indeed, the first copy of L has no marked points and is attached to C' at one point. So the first step in stabilization will prune L reducing the number of special points on C' from 4 to 3.

(ii): In the family defining β_i, only the i^{th} marked point on L varies. After adding the d new marked points, L is a 3-pointed prestable curve; marked by the node p, the i^{th} marked point, and the point q. For every base the only family of genus 0, 3-pointed, stable curves is the constant family. So upon stabilization, this family of genus 0, 3-pointed, stable curves becomes the constant family.

In the family defining γ, only the attachment point of the two copies of L varies. The first copy of L gives a family of 2-pointed, prestable curves; marked by q and the attachment point of the two copies of L. This is unstable. Upon stabilization, the first copy of L is pruned and the marked point q on the first copy is replaced by a marked point on the second copy at the original attachment point. Now the second copy of L gives a family of 3-pointed, prestable curves; marked by the attachment point p of the second and third irreducible components, the attachment point of the first and second components, and q. For the same reason as in the last paragraph, this becomes a constant family.

(iii): Each stable map in $\alpha(\overline{\mathrm{M}}_{0,n+d} \times \mathbb{P}^{r-1})$ is obtained from a genus 0, $(n+d)$-pointed, stable curve $(C_0, (p_1, \dots, p_n, q_1, \dots, q_d))$ and a line L in $\mathbb{P}^r$ containing p by attaching for each $1 \leq i \leq n$, a copy C_i of L to C_0 where p in C_i is identified with q_i in C_0. The map to $\mathbb{P}^r$ contracts C_0 to p, and sends each curve C to L via the identity morphism. Denoting by r the intersection point of L and Π, the inverse image of Π consists of the d points $r_1, \dots, r_d$, where r_i is the copy of r in C_i.

The component C_i is a 2-pointed, prestable curve: marked by the attachment point p of C_i and by r_i. This is unstable. So, upon stabilization, C_i is pruned and the marked point r_i is replaced by a marking on C_0 at the point of attachment of C_0 and C_i, namely q_i. Therefore, up to relabeling of the last d marked points, the result is the genus 0, $(n+d)$-pointed, stable curve we started with, $(C_0,(p_1,\dots,p_n,q_1,\dots,q_d))$. □

In the previous section we constructed a map (see Definition 4.4)

$$v : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}.$$

In this section we prove that the image of v, the divisor classes $\mathcal{H}$, $\mathcal{T}$ and the tautological divisors $\mathcal{L}_i$, generate $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}$.

The divisor class $\mathcal{H}_\Lambda$, [Pa, Prop. 1] is the class of stable maps whose image intersects a fixed codimension 2 linear space Λ of $\mathbb{P}^r$. This is defined to be the empty divisor if $r = 1$. For convenience, assume Λ is contained in Π and does not intersect L or the curves C used to define β_i and γ. If $n \geq 1$, the divisors $\mathcal{L}_{i,\Pi}$, $i = 1,\dots,n$, [Pa, Prop. 1] are the pull-back by ev_i of the Cartier divisor Π. If $d \geq 1$, the last divisor is $\mathcal{T}_\Pi$, [Pa, §2.3]; the divisor of stable maps $(C,(p_1,\dots,p_n),f)$ such that $f^{-1}(\Pi)$ is not a reduced, finite set of degree d. This is defined to be the empty divisor if $d = 1$. In [Pa] Pandharipande proves that $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ and $\mathcal{T}_\Pi$ are irreducible Cartier divisors (when they are nonempty).

Lemma 4.7. (i) *The Cartier divisors $\mathcal{T}_\Pi$, $\mathcal{L}_{i,\Pi}$ and $\mathcal{H}_\Lambda$ are NEF.*

(ii) *The pull-backs $\alpha^*(\mathcal{T}_\Pi)$ and $\alpha^*(\mathcal{L}_{i,\Pi})$ are zero. The pull-back $\alpha^*(\mathcal{H}_\Lambda)$ equals $(0,\mathcal{O}_{\mathbb{P}^{r-1}}(d))$ in $Pic(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times Pic(\mathbb{P}^{r-1})$; if $r=1$, then $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ is the trivial invertible sheaf.*

(iii) *Assume $n \geq 1$ so that β_i is defined for $1 \leq i \leq n$. The pull-backs $\beta_i^*(\mathcal{T}_\Pi)$ and $\beta_i^*(\mathcal{H}_\Pi)$ are zero. For $1 \leq j \leq n$ different from i, $\beta_i^*(\mathcal{L}_{j,\Pi})$ is zero. Finally, $\beta_i^*(\mathcal{L}_{i,\Pi})$ is $\mathcal{O}_{\mathbb{P}^1}(1)$.*

(iv) *Assume $d \geq 2$ so that γ is defined. The pull-backs $\gamma^*(\mathcal{H}_\Lambda)$ and $\gamma^*(\mathcal{L}_{i,\Pi})$ are zero, and $\gamma^*(\mathcal{T}_\Pi)$ is $\mathcal{O}_{\mathbb{P}^1}(2)$ in $Pic(\mathbb{P}^1)$.*

Proof. **(i):** By an argument similar to the one in Lemma 4.5, these divisors are base-point-free (whenever they are non-empty). The divisor $\mathcal{H}_\Lambda$ is big if $r \geq 2$, and $\mathcal{T}_\Pi$ is big if $d \geq 2$. The divisors $\mathcal{L}_i$ are not big.

(ii): By the proof of Lemma 4.6, the image of α is in O_Π, which is disjoint from $\mathcal{T}_\Pi$. Also, $\mathrm{ev}_i \circ \alpha$ is the constant morphism with image p, so the inverse image of $\mathcal{L}_i$ is empty. Finally, the pull-back of $\mathcal{H}_\Pi$ equals the pull-back under the diagonal Δ of the Cartier divisor $\sum_{j=1}^{d} \mathrm{pr}_j^{-1}(\Lambda)$ in $(\mathbb{P}^{r-1})$, where Λ is considered as a divisor in $\mathbb{P}^{r-1}$ via projection from p.

(iii): Since the image of β_i is disjoint from $\mathcal{H}_\Pi$, $\mathcal{T}_\Pi$ and $\mathcal{L}_{j,\Pi}$ for $j \neq i$, the corresponding pull-backs are zero. The map $ev_i \circ \beta_i : \mathbb{P}^1 \to \mathbb{P}^r$ embeds $\mathbb{P}^1$ as the line L in $\mathbb{P}^r$, hence $\beta_i^*(\mathcal{L}_{i,\Pi}) = \mathcal{O}_{\mathbb{P}^1}(1)$.

(iv): Since neither the image curve nor the marked points vary under γ, clearly $\gamma^*\mathcal{H}_\Lambda$ and $\gamma^*\mathcal{L}_{i,\Pi}$ are zero. To compute $\gamma^*\mathcal{T}_\Pi$, use [Pa, Lem 2.3.1]. □

The main observation of this section is the following.

Proposition 4.8. *The $\mathbb{Q}$-vector space $Pic(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}$ is generated by $\mathcal{T}_\Pi$, $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ for $1 \leq i \leq n$, and the image of v.*

Proof. When $r \geq 2$, Pandharipande proves that the classes of the divisors $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ for $1 \leq i \leq n$, and the boundary divisors $\Delta_{(A,d_A),(B,d_B)}$ for $((A,d_A),(B,d_B)) \in \Delta$ generate the $\mathbb{Q}$-vector space $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}$, cf. [Pa, Prop. 1]. The tangency divisor $\mathcal{T}$ can be expressed in terms of $\mathcal{H}$ and the boundary divisors as follows [Pa, Lem 2.3.1]:

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \frac{j(d-j)}{d} \sum_{((A,d_A),(B,d_B)),d_A=j} \Delta_{(A,d_A),(B,d_B)}.$$

From Lemmas 4.7 and 4.6 and by pairing with one-parameter families, we see that

$$v(\tilde{\Delta}_{(A,d_A),(B,d_B)}) = \Delta_{(A,d_A),(B,d_B)}$$

unless $(\#A, d_A)$ or $(\#B, d_B)$ equals one of $(0,2)$ or $(1,1)$.

$$v(\tilde{\Delta}_{(A,d_A),(B,d_B)}) = \frac{1}{2}\,\mathcal{T} + \Delta_{(A,d_A),(B,d_B)}$$

if $(\#A, d_A)$ or $(\#B, d_B)$ equals $(0,2)$. Finally,

$$v(\tilde{\Delta}_{(\{i\},1),(\{i\}^c,d-1)}) = \Delta_{(\{i\},1),(\{i\}^c,d-1)} + \mathcal{L}_{i,\Pi}.$$

Consequently, it follows that the classes of the divisors $\mathcal{H}, \mathcal{T}$, $\mathcal{L}_{i,\Pi}$ and the image of v generate the classes of all the boundary divisors in the Kontsevich moduli space. Hence, they generate $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}$.

We can reduce the case $r = 1$ to the case $r \geq 2$. Because L is disjoint from Λ, there is a unique linear projection

$$\mathrm{pr}_\Lambda : (\mathbb{P}^r - \Lambda) \to L$$

whose restriction to L is the identity. This is a vector bundle over L whose associated sheaf of sections is $\mathcal{O}_L(1)^{\oplus(r-1)}$. Composing a stable map to $(\mathbb{P}^r - \Lambda)$ with pr_Λ gives a stable map to L. This defines a 1-morphism,

$$\overline{\mathcal{M}}_{0,n}(\mathrm{pr}_\Lambda, d) : (\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) - \mathcal{H}_\Lambda) \to \overline{\mathcal{M}}_{0,n}(L,d).$$

This is a vector bundle over $\overline{\mathcal{M}}_{0,n}(L,d)$ whose associated sheaf of sections is the sheaf whose fiber at $(C,(p_1,\dots,p_n),f)$ equals $H^0(C, f^*\mathcal{O}_L(1)^{\oplus(r-1)})$. Thus the pull-back homomorphism,

$$\overline{\mathcal{M}}_{0,n}(\mathrm{pr}_\Lambda, d)^* : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(L,d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) - \mathcal{H}_\Lambda),$$

is an isomorphism, cf. [Ful, Thm. 3.3(a)].

The hyperplane Π is the closure of $\mathrm{pr}_\Lambda^{-1}(L \cap \Pi)$. Thus $U_\Pi - \mathcal{H}_\Lambda \cap U_\Pi$ (see Definition 4.3) is the inverse image of the corresponding open substack of $\overline{\mathcal{M}}_{0,n}(L,d)$ for $L \cap \Pi$ inside L. The inverse image of $\mathcal{T}_{L\cap\Pi}$, resp. $\mathcal{L}_{i,L\cap\Pi}$, equals the restriction of $\mathcal{T}_\Pi$, resp. $\mathcal{L}_{i,\Pi}$. And $\phi_{L\cap\Pi} \circ \overline{\mathcal{M}}_{0,n}(\mathrm{pr}_\Lambda, d)$ equals the restriction of ϕ_Π. Thus $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) - \mathcal{H}_\Lambda) \otimes \mathbb{Q}$ is generated by $\mathcal{T}_\Pi$, $\mathcal{L}_{i,\Pi}$ for $1 \leq i \leq n$, and the image of v if and only if the same is true for $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,d)) \otimes \mathbb{Q}$.

□

Proof of Theorem 4.1. Now we can complete the proof of Theorem 4.1. Denote by

$$\widetilde{v} : P_{r,n,d} \otimes \mathbb{Q} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}$$

the unique homomorphism whose restriction to $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$ is v (see Definition 4.4) , whose restriction to $\mathrm{Pic}(\mathbb{P}^{r-1})$ sends $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ to $[\mathcal{H}_\Lambda]$, whose restriction to the i^{th} factor of $\mathrm{Pic}(\mathbb{P}^1)^n$ sends $\mathcal{O}_{\mathbb{P}^1}(1)$ to $[\mathcal{L}_i]$ if $n \geq 1$, and whose restriction to the last factor $\mathrm{Pic}(\mathbb{P}^1)$ (assuming $d \geq 2$) sends $\mathcal{O}_{\mathbb{P}^1}(1)$ to $1/2\ [\mathcal{T}_\Pi]$. By Lemma 4.6 (ii), (iii) and by Lemma 4.7, $u \otimes \mathbb{Q} \circ \widetilde{v}$ is the identity map. In particular, $\widetilde{v}$ is injective. By Proposition 4.8, $\widetilde{v}$ is surjective. Thus $\widetilde{v}$ and $u \otimes \mathbb{Q}$ are isomorphisms.

Because α, β_i and γ are morphisms, for every NEF, resp. eventually free, divisor D in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$, $\alpha^*(D)$, $\beta_i^*(D)$, and $\gamma^*(D)$ are NEF, resp. eventually free. Denote,

$$D_1 = \alpha_1^*(D), \quad a\ [\mathcal{O}_{\mathbb{P}^{r-1}}(1)] = \alpha_2^*(D), \quad b_i\ [\mathcal{O}_{\mathbb{P}^1}(1)] = \beta_i^*(D), \quad c\ [\mathcal{O}_{\mathbb{P}^1}(1)] = \gamma^*(D),$$

where by convention a is defined to be 0 if $r = 1$ and c is defined to be 0 if $d = 1$. If D is NEF, resp. eventually free, D_1 is NEF, resp. eventually free, in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, and $a, b_i, c \geq 0$.

Conversely, by Lemma 4.5, for every NEF, resp. eventually free, divisor D_1 in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $v(D_1)$ is NEF, resp. eventually free. By Lemma 4.7(i), for $a, b_i, c \geq 0$, $a[\mathcal{H}_\Lambda]$, $b_i[\mathcal{L}_{i,\Pi}]$ and $c/2\ [\mathcal{T}_\Pi]$ are NEF and eventually free. Since a sum of NEF, resp. eventually free, divisors is NEF, resp. eventually free, $D = v(D_1) + a\ [\mathcal{H}_\Lambda] + b_i\ [\mathcal{L}_i] + c/2\ [\mathcal{T}_\Pi]$ is NEF, resp. eventually free. Therefore D is NEF if and only if $u \otimes \mathbb{Q}(D)$ is in the product of the NEF cones of the factors. This argument needs to be modified in the obvious way when $(n, d) = (0, 3)$ and $(1, 2)$ to account for the slight variations in the formulae.

Because the interior of a product of cones equals the product of the interiors of the cones, by Kleiman's criterion, D is ample iff $u \otimes \mathbb{Q}(D)$ is contained in the product of the ample cones of the factors. $\square$

Theorem 4.1 has the following important corollary.

Theorem 4.9. *For every integer $r \geq 1$ and $d \geq 2$, there is a contraction,*

$$cont: \overline{M}_{0,0}(\mathbb{P}^r, d) \to Y,$$

restricting to an open immersion on the interior $M_{0,0}(\mathbb{P}^r, d)$ and whose restriction to the boundary divisor $\Delta_{k,d-k} \cong M_{0,1}(\mathbb{P}^r, k) \times_{\mathbb{P}^r} M_{0,1}(\mathbb{P}^r, d-k)$ factors through the projection to $\overline{M}_{0,1}(\mathbb{P}^r, d-k)$ for each $1 \leq k \leq \lfloor d/2 \rfloor$. The following divisor is the pullback of an ample divisor on Y,

$$D_{r,d} = \mathcal{T} + \sum_{k=2}^{\lfloor d/2 \rfloor} k(k-1)\Delta_{k,d-k}.$$

Theorem 4.9 has implications for the study of rational curves on Fano manifolds. For instance J. Starr has proved the following nice consequence.

4.2. **The effective cone of the Kontsevich moduli space.** The main problem we would like to address is the following:

Problem 4.10. Describe the cone of effective divisor classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ in terms of the standard generators of the Picard group.

Denote by P_d the $\mathbb{Q}$-vector space of dimension $\lfloor d/2 \rfloor + 1$ with basis labeled $\mathcal{H}$ and $\Delta_{k,d-k}$ for $k = 1, \ldots, \lfloor d/2 \rfloor$. For each $r \geq 2$, there is a $\mathbb{Q}$-linear map

$$u_{d,r} : P_d \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$$

that is an isomorphism of $\mathbb{Q}$-vector spaces.

Definition 4.11. For every integer $r \geq 2$, denote by $\mathrm{Eff}_{d,r} \subset P_d$ the inverse image under $u_{d,r}$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

Proposition 4.12. *For every integer $r \geq 2$, $\mathit{Eff}_{d,r}$ is contained in $\mathit{Eff}_{d,r+1}$. For every integer $r \geq d$, $\mathit{Eff}_{d,r}$ equals $\mathit{Eff}_{d,d}$.*

Proof of Proposition 4.12. Let $p \in \mathbb{P}^{r+1}$ be a point, denote $U = \mathbb{P}^{r+1} - \{p\}$, and let $\pi : U \to \mathbb{P}^r$ be a linear projection from p. This induces a smooth 1-morphism

$$\overline{\mathcal{M}}_{0,0}(\pi, d) : \overline{\mathcal{M}}_{0,0}(U, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d).$$

Let $i : U \to \mathbb{P}^{r+1}$ be the open immersion. This induces a 1-morphism

$$\overline{\mathcal{M}}_{0,0}(i, d) : \overline{\mathcal{M}}_{0,0}(U, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)$$

relatively representable by open immersions. The complement of the image of $\overline{\mathcal{M}}_{0,0}(i, d)$ has codimension r, which is greater than 2. Therefore, the pull-back morphism

$$\overline{\mathcal{M}}_{0,0}(i, d)^* : \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(U, d))$$

is an isomorphism. So there is a unique homomorphism

$$h : \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d))$$

such that

$$\overline{\mathcal{M}}_{0,0}(\pi, d)^* = \overline{\mathcal{M}}_{0,0}(i, d)^* \circ h.$$

Recalling from the introduction that $u(r, d)$ is the map that identifies the Picard group of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d))$ with the vector space spanned by $\mathcal{H}$ and the boundary divisors $\Delta_{k,d-k}$, we see that $h \circ u_{d,r}$ equals $u_{d,r+1}$. So to prove $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$, it suffices to prove that $\overline{\mathcal{M}}_{0,0}(\pi, d)$ pulls back effective divisors to effective divisors classes, which follows since $\overline{\mathcal{M}}_{0,0}(\pi, d)$ is smooth.

Next assume $r \geq d$. Let D be any effective divisor in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. A general point in the complement of D parameterizes a stable map $f : C \to \mathbb{P}^r$ such that $f(C)$ spans a d-plane. Denote by $j : \mathbb{P}^d \to \mathbb{P}^r$ a linear embedding whose image is this d-plane. There is an induced 1-morphism

$$\overline{\mathcal{M}}_{0,0}(j, d) : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d).$$

The map $\overline{\mathcal{M}}_{0,0}(j, d)^* \circ u_{d,r}$ equals $u_{d,d}$. By construction, $\overline{\mathcal{M}}_{0,0}(j, d)^*([D])$ is the class of the effective divisor $\overline{\mathcal{M}}_{0,0}(j, d)^{-1}(D)$, i.e., $[D]$ is in $\mathrm{Eff}_{d,d}$. Thus $\mathrm{Eff}_{d,d}$ contains $\mathrm{Eff}_{d,r}$, which in turn contains $\mathrm{Eff}_{d,d}$ by the last paragraph. Therefore $\mathrm{Eff}_{d,r}$ equals $\mathrm{Eff}_{d,d}$. □

In view of Proposition 4.12 it is especially interesting to understand $\mathrm{Eff}_{d,d}$. We will concentrate on this case.

When $r = d$, the locus parameterizing stable maps $f : C \to \mathbb{P}^d$ of degree d whose set theoretic image does not span $\mathbb{P}^d$. We will denote its class by D_{deg}. The class is easily calculated in terms of the standard divisors.

Lemma 4.13. *The class D_{deg} equals*

$$D_{deg} = \frac{1}{2d}\left[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}\right]. \tag{2}$$

Proof. We will prove the equality (2) by intersecting $D_{\rm deg}$ by test curves. Fix a general rational normal scroll of degree i and a general rational normal curve of degree $d-i-1$ intersecting the scroll in one point p. Consider the one-parameter family C_i of degree d curves consisting of the fixed degree $d-i-1$ rational normal curve union curves in a general pencil (that has p as a base-point) of degree $i+1$ rational normal curves on the scroll. When $2 \leq i \leq \lfloor d/2 \rfloor$, C_i has the following intersection numbers with $\mathcal{H}$ and $D_{\rm deg}$.

$$C_i \cdot \mathcal{H} = i, \quad C_i \cdot D_{\rm deg} = 0.$$

The curve C_i is contained in the boundary divisor $\Delta_{i+1,d-i-1}$ and has intersection number

$$C_i \cdot \Delta_{i+1,d-i-1} = -1$$

with it. The intersection number of C_i with the boundary divisors $\Delta_{i,d-i}$ and $\Delta_{1,d-1}$ is non-zero and given as follows:

$$C_i \cdot \Delta_{i,d-i} = 1, \quad C_i \cdot \Delta_{1,d-1} = i+1.$$

Finally, the intersection number of C_i with all the other boundary divisors is zero. When $i = 1$, we have to modify the intersection number of C_1 with $\Delta_{1,d-1}$ to read $C_1 \cdot \Delta_{1,d-1} = 3$. Next consider the one-parameter family B_1 of rational curves of degree d that contain $d+2$ general points and intersect a general line. The intersection number of B_1 with all the boundary divisors but $\Delta_{1,d-1}$ is zero. Clearly $B_1 \cdot D_{\rm deg} = 0$. By the algorithm for counting rational curves in projective space given in [V] it follows that

$$B_1 \cdot \mathcal{H} = \frac{d^2+d-2}{2}, \quad B_1 \cdot \Delta_{1,d-1} = \frac{(d+2)(d+1)}{2}.$$

This determines the class of $D_{\rm deg}$ up to a constant multiple. In order to determine the multiple, consider the curve C that consists of a fixed degree $d-1$ curve and a pencil of lines in a general plane intersecting the curve in one point. The curve C has intersection number zero with all the boundary divisors but $\Delta_{1,d-1}$ and has the following intersection numbers:

$$C \cdot \mathcal{H} = 1, \quad C \cdot D_{\rm deg} = 1, \quad C \cdot \Delta_{1,d-1} = -1.$$

The lemma follows from these intersection numbers. □

$D_{\rm deg}$ plays a crucial role in describing the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. The following theorem completely describes the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$.

Theorem 4.14. *The class of a divisor lies in the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ if and only if it is a non-negative linear combination of the class of D_{deg} and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.*

Following Keel one may reduce the proof of this theorem to determining the effective cone of $\overline{M}_{0,d}/\mathfrak{S}_d$. However, this proof is not significantly simpler and has the disadvantage that it does not generalize to other contexts. We will therefore give a better proof.

Proof. Since $D_{\rm deg}$ and the boundary divisors are effective, any non-negative rational linear combination of these divisors lies in the effective cone. The main content of the theorem is to show that there are no other effective divisor classes.

Definition 4.15. A reduced, irreducible curve C on a scheme X is a *moving curve* if the deformations of C cover a Zariski open subset of X. More precisely, a curve C is a moving curve if there exists a flat family of curves $\pi : \mathcal{C} \to T$ on X such that $\pi^{-1}(t_0) = C$ for $t_0 \in T$ and for a Zariski open subset $U \subset X$ every point $x \in U$ is contained in $\pi^{-1}(t)$ for some $t \in T$. We call the class of a moving curve a *moving curve class.*

An obvious observation is that the intersection pairing between the class of an effective divisor and a moving curve class is always non-negative. Intersecting divisors with a moving curve class gives an inequality for the coefficients of an effective divisor class. The strategy for the proof of Theorem 4.14 is to produce enough moving curves to force the effective divisor classes to be a non-negative linear combination of $D_{\deg}$ and the boundary classes.

Lemma 4.16. *If $C \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is a reduced, irreducible curve that intersects the complement in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, then C is a moving curve.*

Proof. The automorphism group of $\mathbb{P}^d$ acts transitively on rational normal curves. An irreducible curve of degree d that spans $\mathbb{P}^d$ is a rational normal curve. Hence, a curve $C \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ that intersects the complement in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, contains a point that represents a map that is an embedding of $\mathbb{P}^1$ as a rational normal curve. The translations of C by $\mathbb{P}GL(d+1)$ cover a Zariski open set of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. □

First, observe that if D is an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ and D has the class

$$a\mathcal{H} + \sum_{k=1}^{\lfloor d/2 \rfloor} b_{k,d-k} \Delta_{k,d-k},$$

then $a \geq 0$. Furthermore, if $a = 0$, then $b_{k,d-k} \geq 0$. Consider a general projection of the d-th Veronese embedding of $\mathbb{P}^2$ to $\mathbb{P}^d$. Consider the image of a pencil of lines in $\mathbb{P}^2$. By Lemma 4.16 this is a moving one-parameter family C of degree d rational curves that has intersection number zero with the boundary divisors. It follows from the inequality $C \cdot D \geq 0$ that $a \geq 0$.

Furthermore, suppose that $a = 0$. Consider a general pencil of $(1,1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Take a general projection to $\mathbb{P}^d$ of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ by the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(i, d-i)$. By Lemma 4.16 the image of the pencil gives a moving one-parameter family C of degree d curves whose intersection with $\Delta_{k,d-k}$ is zero unless $k = i$. The relation $C \cdot D \geq 0$ implies that if $a = 0$, then $b_{i,d-i} \geq 0$. We conclude that Theorem 4.14 is true if $a = 0$. We can, therefore, assume that $a > 0$.

Suppose that for every $1 \leq i \leq \lfloor d/2 \rfloor$, we could construct a moving curve C_i in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ with the property that $C_i \cdot \Delta_{k,d-k} = 0$ for $k \neq i$ and that the ratio of $C_i \cdot \Delta_{i,d-i}$ to $C_i \cdot \mathcal{H}$ is given by

$$\frac{C_i \cdot \Delta_{i,d-i}}{C_i \cdot \mathcal{H}} = \frac{d+1}{i(d-i)}. \tag{3}$$

Observe that given these intersection numbers, Lemma 4.13 implies that $C_i \cdot D_{\deg} = 0$. Theorem 4.14 follows from the inequalities $C_i \cdot D \geq 0$.

We now construct approximations to these curves.

Proposition 4.17. *Let k, j and d be positive integers subject to the condition that $2k \leq d$. There exists an integer $n(k,d)$ depending only on k and d such that the linear system*

$$L'(j) = d\ F_1 + \left(\frac{jk(k+1)}{2} - 1\right)\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}$ general points is non-special for every $j >> 0$. The integer $n(k,d)$ may be taken to be

$$n(k,d) = \lceil 2(d+1)/k \rceil.$$

Proposition 4.17 implies Theorem 4.14. As in the previous subsection we consider the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in

$$j(d+1) + \frac{n(k,d)(k-1)(k+2)}{2}$$

general points. The proper transform of the fibers F_2 under the linear system

$$d\ F_1 + \frac{jk(k+1)}{2}\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

gives a one-parameter family $C_k(j)$ of rational curves of degree d that has intersection number zero with D_{deg}. Letting j tend to infinity we obtain a sequence of moving curves $C_k(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ that has intersection zero with all the boundary divisors but $\Delta_{1,d-1}$ and $\Delta_{k,d-k}$. Unfortunately, the intersection of $C_k(j)$ with $\Delta_{1,d-1}$ is not zero and the ratio of $C_k(j) \cdot \mathcal{H}$ to $C_k(j) \cdot \Delta_{k,d-k}$ is not the one required by Equation (3). However, as j tends to infinity, the ratio of the intersection numbers $C_k(j) \cdot \Delta_{1,d-1}$ to $C_k(j) \cdot \mathcal{H}$ tends to zero and the ratio of $C_k(j) \cdot \Delta_{k,d-k}$ to $C_k(j) \cdot \mathcal{H}$ tends to the desired ratio $\frac{d+1}{k(d-k)}$. Theorem 4.14 follows.

Proof of Proposition 4.17. The specialization technique in §2 of [Ya] yields the proof of the proposition. We will specialize the points of multiplicity k one by one onto a point q. At each stage the k-fold point that we specialize will be in general position. We will first slide the point along a fiber f_1 in the class F_1 onto the fiber f_2 in the fiber class F_2 containing the point q. We then slide the point onto q along f_2. We will record the flat limit of this degeneration.

There is a simple checker game that describes the limits of these degenerations. This checker game for $\mathbb{P}^2$ is described in §2 of [Ya]. The details for $\mathbb{P}^1 \times \mathbb{P}^1$ are identical. The global sections of the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)$ are bi-homogeneous polynomials of bi-degree a and b in the variables x, y and z, w, respectively. A basis for the space of global sections is given by $x^i y^{a-i} z^j w^{b-j}$, where $0 \leq i \leq a$ and $0 \leq j \leq b$. We can record these monomials in a rectangular $(a+1) \times (b+1)$ grid. In this grid the box in the i-th row and the j-th column corresponds to the monomial $x^i y^{a-i} z^j w^{b-j}$.

If we impose an ordinary k-fold point on the linear system at $([x:y],[z:w]) = ([0:1],[0:1])$, then the coefficients of the monomials

$$y^a w^b, xy^{a-1}w^b, \ldots, x^{k-1}y^{a-k+1}z^{k-1}w^{b-k+1}$$

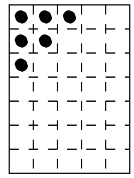

FIGURE 8. Imposing a triple point on $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,6)$.

must vanish. We depict this by filling in a $k \times k$ triangle of checkers into the boxes at the upper left hand corner as in Figure 8. The coefficients of the monomials represented by boxes that have checkers in them must vanish.

We first slide the k-fold point along the fiber f_1 onto the point $([x:y],[z:w]) = ([1:0],[0:1])$. This correspond to the degeneration

$$([x:y],[z:w]) \mapsto ([x:ty],[z:w]).$$

The flat limit of this degeneration is described by the vanishing of the coefficients of certain monomials (assuming none of the checkers fall out of the rectangle). The monomials whose coefficients must vanish are those that correspond to boxes with checkers in them when we let the checkers fall according to the force of gravity. The first two panels in Figure 9 depict the result of applying this procedure to a 4-fold point when there is an aligned ideal condition at the point $([x:y],[z:w]) = ([1:0],[1:0])$.

We then follow this degeneration with a degeneration that specializes the k-fold point to q by sliding along the fiber f_2. This degeneration is explicitly given by

$$([x:y],[z:w]) \mapsto ([x:y],[z:tw]).$$

The flat limit is described by the vanishing of the coefficients of the monomials that have checkers in them when we slide all the checkers as far right as possible. The last two panels of Figure 9 depict this degeneration.

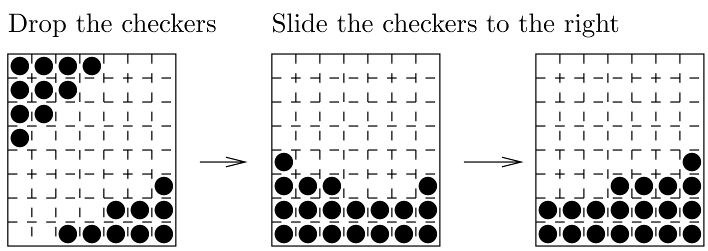

FIGURE 9. Depicting the degenerations by checkers.

S. Yang proves that, provided none of the checkers fall out of the ambient rectangle during these moves, these checker movements do correspond to the flat limits of the linear systems under the given degenerations. If one can play this checker game with all the multiple points that one imposes on a linear system so that during the game none of the checkers fall out of the rectangle, one can conclude that the multiple points impose independent conditions on the linear system. The limit linear system has the expected dimension. In particular, it is non-special. By upper semi-continuity the original linear system must also have the expected dimension and be non-special. Unfortunately, when one plays this game, occasionally checkers may fall out of the rectangle. In that case we lose information on what the limits are. This may happen even if the original linear system has the expected dimension.

In order to conclude the proposition we need to show that if we impose at most $j(d+1)-n(k,d)$ points of multiplicity k on the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d, jk(k+1)/2)$ where $2k \leq d$, we do not lose any checkers when we specialize all the k-fold points by the degeneration just described. This suffices to conclude the proposition because general simple points always impose independent conditions.

The main observation is that if there is a safety net of empty boxes at the top of the rectangle, then the checkers will not fall out of the box. The proof of the proposition is completed by noting the following simple facts.

(1) At any stage of the degeneration the height of the checkers in the rectangle is at most k larger than the highest row full of checkers.
(2) The left most checker of a row is to the lower left of the left most checker of any row above it.

If there are at least $(k+1)(d+1)$ empty boxes in our rectangle, then by the above two observations when we specialize a k-fold point we do not lose any of the checkers. As long as $n(k,d) \geq \lceil 2(d+1)/k \rceil$, there is always at least $(k+1)(d+1)$ boxes empty. Hence until the stage where we specialize the last k-fold point we cannot lose any checkers. This concludes the proof. $\square$

This also concludes the proof of the theorem. $\square$

Remark 4.18. We observe that both Theorem 4.1 and Theorem 4.14 admit generalizations to other homogeneous targets. These may be proved using the methods developed here.

Exercise 4.19. Determine the ample and stable effective cones of the Kontsevich moduli space of stable maps into Grassmannians.

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