18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 22-23

LECTURES: ABHINAV KUMAR

1. Classification (contd.)

Recall the classification from before:

(a) $\exists$ an integral curve C on X s.t. $K \cdot C < 0$.
(b) $K \equiv 0$.
(c) $K^2 = 0$, $K \cdot C \geq 0$ for all integral curves C, and $\exists C'$ s.t. $K \cdot C' > 0$.
(d) $K^2 > 0$, and $K \cdot C \geq 0$ for all integral curves C.

Theorem 1. *Let X be a minimal surface in class (b) or (c). Then $p_4 > 0$ or $p_6 > 0$. So, if X belongs to class (b), then $4K \sim 0$ or $6K \sim 0$, and if X belongs to class (c), then $|4K|$ or $|6K|$ contains a strictly positive divisor.*

Proof. If X is in class (b) or (c), then $K^2 = 0, K \cdot C \geq 0$ for any integral curve C. Thus, either $2K \sim 0$ or X has an elliptic/quasielliptic fibration. If $2K \sim 0$, then of course $K \equiv 0$, implying that X is in class (b) and $p_2 \neq 0 \implies p_4, p_6 \neq 0$ as well. Otherwise, let $f : X \to B$ be the stated fibration, and assume that $p_g = 0$ (if $p_g = p_1 > 0$, then $p_n > 0$ for all $n \geq 2$ and the theorem holds). Now, for X minimal with $K^2 = 0, p_g = 0$, we have that $p_g = h^0(B, L^{-1} \otimes \omega_B) = 0$ and $\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T)$. But $\chi(\mathcal{O}_X) \geq 0$ from the list of last time, so by Riemann-Roch, $p_a(B) = 1, \chi(\mathcal{O}_X) = 0, \ell(T) = 0$ or $p_a(B) = 0, \chi(\mathcal{O}_X) + \ell(T) < 2$. We analyze these two cases separately.

Case 1: having no exceptional fibers implies that $a_i = m_i - 1$ for all i. If $\exists$ a multiple fiber $m_i P_i$ with $m_i \geq 2$, then (say $m_1 \geq 2$)

$$(1)\quad \begin{aligned} \omega_X &= f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X\Big(\sum (m_i - 1)P_i\Big) \\ \omega_X^2 &= f^*(L^{-2} \otimes \omega_B^2) \otimes \mathcal{O}_X\Big(\sum_{i>1} (2m_i - 2)P_i\Big) \otimes f^*\mathcal{O}_B(b_1) \otimes \mathcal{O}_X((m_1 - 2)P_1) \end{aligned}$$

Since $\deg(L^{-2} \otimes \omega_B^2 \otimes \mathcal{O}_B(b_1) \geq 1$ and B is an elliptic curve, $p_2 \geq 1$ and so $p_4, p_6 > 0$, proving the theorem. If f has no multiple fibers, then $\omega_X = f^*(L^{-1} \otimes \omega_B)$, and $\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) = 0 \implies \omega_X \equiv 0$ and thus X is in class (b). It also must be case 5 from last time, thus giving a bielliptic surface and another elliptic fibration $g : X \to \mathbb{P}^1$, placing it in case 2 of our analysis.

Case 2: $p_a(B) = 0$, i.e. $B = \mathbb{P}^1, p_g = 0$, i.e. $\deg(L^{-1} \otimes \omega_B) = -2 + \chi(\mathcal{O}_X) + \ell(T) < 0$. Now,

$$\lambda(f) = -2 + \chi(\mathcal{O}_X) + \ell(T) + \sum \frac{a_i}{m_i} \geq 0 \tag{2}$$

and an easy check gives

$$h^0(nK) = 1 + n\left(-2 + \chi(\mathcal{O}_X) + \ell(T)\right) + \sum \left\lfloor \frac{na_i}{m_i} \right\rfloor \tag{3}$$

Case 2A: $\ell(T) = 0$, so $a_i = m_i - 1$ for all i. Also, $-2 + \chi(\mathcal{O}_X) + \ell(T) < 0$ gives $\chi(\mathcal{O}_X) = 0, 1$. If $\chi(\mathcal{O}_X) = 0$, then we must have at least 3 multiple fibers (because we need $-2 + \sum \frac{m_i - 1}{m_i} \geq 0$.
- ≥ 4 multiple fibers, $m_i \geq 2$. Then check $|2K| \neq \varnothing$, i.e. $p_2 > 0$.
- 3 multiple fibers, all $m_i \geq 3$. Then $|3K| \neq \varnothing$, so $p_3, p_6 > 0$.
- 3 multiple fibers, $m_1 = 2, m_2, m_3 \geq 4 \implies |4K| \neq \varnothing$.
- 3 multiple fibers, $m_1 = 2, m_2 = 3, m_3 \geq 6 \implies |6K| \neq \varnothing$.

Case 2B: $\ell(T) = 1$, i.e. there is exactly one wild/exceptional fiber F. Now $-2 + \chi(\mathcal{O}_X) + \ell(T) < 0 \implies \chi(\mathcal{O}_X) = 0$. Also, $p_g = 0 \implies q = 1$. Applying the corollary from the previous lecture, we get $a_1 = m_1 - 1$ or $m_1 - 1 - \alpha_1$, where α_1 is a common divisor of m_1 and $a_1 + 1$. Since $-1 + \sum \frac{a_i}{m_i} \geq 0$, there are at least 2 multiple fibers, so either
- There exist at least 2 multiple curves with $a_i = m_i - 1$, so $|2K| \neq \varnothing$.
- F has $m_1 = 3, a_1 = 1, \alpha_1 = 1, m_2 \geq 3 \implies |3K| \neq \varnothing$.
- F has $m_1 = 4, a_1 = 1, \alpha_1 = 2, m_2 \geq 4 \implies |4K| \neq \varnothing$.
- F has $m_1 = \beta_1 \alpha_1, \beta_1 \geq 4 \implies \frac{a_1}{m_1} \geq \frac{1}{2} \implies |2K| \neq \varnothing$.
- F has $m_1 = 2\alpha_1, a_1 = \alpha_1 - 1, \alpha_1 \geq 3, m_2 \geq 3 \implies |3K| \neq \varnothing$.
- F has $m_1 = 3\alpha_1, a_1 = 2\alpha_1 - 1, \alpha_1 \geq 2, m_2 \geq 2 \implies |2K| \neq \varnothing$.

This concludes the proof. □

So for X a minimal surface with elliptic/quasielliptic fibration $f : X \to B$,
- X is in (a) $\Leftrightarrow \lambda(f) < 0 \Leftrightarrow \kappa(X) = -\infty \Leftrightarrow p_n = 1 \forall n \geq 1 \Leftrightarrow p_4 = p_6 = 0 \Leftrightarrow p_{12} = 0$,
- X is in (b) $\Leftrightarrow \lambda(f) = 0 \Leftrightarrow \kappa(X) = 0 \Leftrightarrow nK \sim 0$ for some $n \geq 1 \Leftrightarrow 4K \sim 0$ or $6K \sim 0 \Leftrightarrow 12K \sim 0$,
- X is in (c) $\Leftrightarrow \lambda(f) > 0 \Leftrightarrow \kappa(X) = 1 \Leftrightarrow nK$ has a strictly positive divisor for some $n \geq 1 \Leftrightarrow |12K|$ has a strictly positive divisor.

Theorem 2. *Let X be a minimal surface in class (d), i.e. $K^2 > 0, K \cdot C \geq 0$ for all curves C on X. Then $|2K| \neq \varnothing$, and for sufficiently large n, the linear system $|nK|$ is free of base points and defines a morphism $\phi_n = \phi_{|nK|} : X \to \mathbb{P}(H^0(\mathcal{O}_X(nK))^\vee)$ s.t.*
- *$X_n = \phi_n(X)$ is normal, with at most rational double points as singularities, i.e. desingularizing gives a fixed cycle Z (the smallest divisor*

with support in the exceptional locus with $Z \cdot E_i \leq 0 \,\forall\, i$) which satisfies $p_a(Z) = 0, Z^2 = -2$.

- *ϕ_n is an isomorphism away from the singular locus, i.e.*

$$X \smallsetminus \phi_n^{-1}(\mathrm{Sing}(X_n)) \xrightarrow{\sim} X_n \smallsetminus \mathrm{Sing}(X_n) \tag{4}$$

In this case, $\kappa(X) = 2$.

Proof. Exercise. Use Riemann-Roch, the Hodge index theorem, and Nakai-Moishezon. The point is that if $K \cdot C > 0$ for all integral curves C on X, then K is ample and we're through. So the problem comes from curves E_i s.t. $K \cdot E_i = 0$. But $K^2 > 0$ and on the orthogonal complement of K, the form $(\cdot)$ is negative definite, so there at most finitely such curves E_i. Show that they are rational ($p_a(E_i) = 0$), and satisfy the criteria of rational double points. □

There are some things left to prove in classification: one is that every minimal surface with $\kappa(X) = 0, b_2 = 6$ is Abelian (see Bombieri-Mumford).

Remark. For a surface of general type, $|nK|$ is base point free for $n \geq 4$, and for $n \geq 5$, $\phi_{|nK|}$ is an isomorphism away from the union of finitely many rational curves.

To review, we have shown that, if X is a minimal surface and $f : X \to B$ is an elliptic/quasi-elliptic fibration, then

- X is in (a) iff $\lambda(f) < 0 \Leftrightarrow \kappa(X) = -\infty \Leftrightarrow |4K| = |6K| = \varnothing$,
- X is in (b) iff $\lambda(f) = 0 \Leftrightarrow \kappa(X) = 0 \Leftrightarrow 4K \sim 0$ or $6K \sim 0$,
- X is in (c) iff $\lambda(f) > 0 \Leftrightarrow \kappa(X) = 1 \Leftrightarrow |4K|$ or $|6K|$ contain strictly positive divisors, and $K^2 = 0$.
- X is in class (d) $\Leftrightarrow \kappa(X) = 2$, and in this case $|2K| = \varnothing$.

Since (a),…,(d) are mutually disjoint and exhaustive, we get the following theorem:

Theorem 3. *Let X be a minimal surface. Then:*

- *$\exists C > 0$ on X s.t. $K \cdot C < 0 \Leftrightarrow \kappa(X) = -\infty \Leftrightarrow p_4 = p_6 = 0 \Leftrightarrow p_{12} = 0 \Leftrightarrow X$ is ruled.*
- *$K \equiv 0 \Leftrightarrow \kappa(X) = 0 \Leftrightarrow 4K \sim 0$ or $6K \sim 0 \Leftrightarrow 12K \sim 0 \Leftrightarrow X$ is abelian, K3, Enriques, or bielliptic,*
- *$K^2 = 0, K \cdot C \geq 0 \,\forall\, C$ integral, and $K \cdot C' > 0$ for some $C \Leftrightarrow \kappa(X) = 1 \Leftrightarrow K^2 = 0$ and $|4K|$ or $|6K|$ contain strictly positive divisors $\Leftrightarrow X$ is honestly elliptic.*
- *$K^2 > 0, K \cdot C \geq 0 \,\forall\, C > 0 \Leftrightarrow \kappa(X) = 2$, and in this case $|2K| = \varnothing$. Such a surface is one of* general type.

Example. Examples of general type surfaces:

(1) $C_1 \times C_2$ where C_1, C_2 are smooth projective curves of genus ≥ 2.

(2) Smooth, complete intersections cut out by $f_1, \dots, f_{n-2}$ in $\mathbb{P}^n$ of degree $d_1, \dots, d_{n-2}$ s.t. $\sum d_i > n+1$ (then $\omega_X = \mathcal{O}_X(\sum d_i - n - 1)$ is ample). Such a surface also has $q = 0$.

(3) Godeaux surface, i.e. a quotient of a Fermat surface. For instance, $x^5 + y^5 + z^5 + t^5 = 0$ in $\mathbb{P}^3$ quotiented by $\mathbb{Z}/5\mathbb{Z}$ acting as

$$\sigma(z, y, z, t) = (x, \zeta y, \zeta^2 z, \zeta^3 t) \tag{5}$$

for a generator ζ of $\mathbb{Z}/5\mathbb{Z}$. There are no fixed point, and the quotient as $q = p_g = 0$ but not $p_2 = 0$, and it is of general type ($K^2 = \frac{1}{5}5 = 1$).

Remark. Here are various results for surfaces we did not cover here:

- Castelnuovo's inequality: for a non-ruled surface X, $\chi_{\text{top}}(X) \geq 0, \chi(\mathcal{O}_X) \geq 0$. If X is of general type, then $\chi(\mathcal{O}_X) > 0, \chi_{\text{top}}(X) \geq 0$ is equivalent to $K^2 \leq 12\chi(\mathcal{O}_X)$.
- Bogomolov-Miyaoka-Yau inequality (conjectured by the first two, proved by Yau): $K^2 \leq 9\chi(\mathcal{O}_X)$.
- Persson: All values for $K^2 \leq 8\chi$ actually occur.
- Mumford, Hirzebruch: $\exists$ surfaces for which $K^2 = 9\chi(\mathcal{O}_X)$.

2. Moduli

First, in the case of $\kappa = -\infty$, we have surfaces ruled over a curve C, whose moduli we can study by studying vector bundles of rank 2 on C. One finds that for $g \geq 2$, there are $3g - 3$ moduli (coming from stable vector bundles). Over $\mathbb{P}^1$, the moduli are a discrete set of points (the Hirzebruch surfaces $\mathbb{F}_n$ or Σ_n). Over an elliptic curve, any rank 2 vector bundle is either decomposable, the extension of $\mathcal{O}_C$ by $\mathcal{O}_C$, or the extension of $\mathcal{O}_C(p)$ by $\mathcal{O}_C$ for $p \in E$.

Next, for $\kappa = 0$, we have different moduli depending on the subcategorization. For abelian surfaces with polarization, there exist Siegel modular varieties coming from quotienting by a discrete subgroup of the symplectic group. For K3 surfaces (marked and with polarization), one obtains a moduli space as an open subset of a quadric in $\mathbb{CP}^{20}$ (a bounded, symmetric domain of type IV). Moding out by the marking, one obtains a quotient by a discrete subgroup of the orthogonal. Similarly, one obtains algebraic varieties characterising the moduli for Enriques surfaces. Lastly, for hyperelliptic and bielliptic surfaces, our previous explicit description allows one to describe the moduli relatively easily.

Then, for honestly elliptic surfaces, one can describe the moduli either by functional and homological invariants, or by the Weierstrass equations.

For surfaces of general type, however, little is known.