18.726 Algebraic Geometry
Spring 2009

Using Čech cohomology, we now give Serre's computation of the cohomology of the twisting sheaves on projective space, and deduce from it Serre's finiteness theorem. See also Hartshorne III.5 (except that you can ignore the last paragraph at the bottom of page 227, which is irrelevant).

1 Finitely generated sheaves

Let $(X, \mathcal{O}_X)$ be a locally ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *finitely generated* if for each point $x \in X$, there exist an open subset U of X containing x, a nonnegative integer n, and a surjection $\mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$ of $\mathcal{O}_X$-modules. (In other words, $\mathcal{F}$ is locally generated by finite collections of sections.)

Lemma. *Let A be a ring and let M be an A-module. Then M is finite as an A-module if and only if the quasicoherent sheaf $\tilde{M}$ is finitely generated.*

Proof. It is clear that finiteness of M implies that $\tilde{M}$ is finitely generated. Conversely, suppose $\tilde{M}$ is finitely generated. We can then cover X with distinguished opens $X_i = D(f_i)$ on each of which $\tilde{M}|_{X_i} = \tilde{M}_{f_i}$ is generated by finitely many sections $m_{ij}/f_i^{h_j}$. Since X is quasicompact, we need only finitely many i, and the resulting m_{ij} generate M because they do so stalkwise. $\square$

Beware that Hartshorne says a sheaf is *coherent* if it is quasicoherent and finitely generated. This is not the correct definition, but it does agree with the correct definition on a *locally noetherian* scheme, which is the only place he ever uses it. I'll give the proper definition of a coherent sheaf later.

2 Odds and ends

We will make frequent use of the following fact.

Lemma. *Let $f : Z \to X$ be a closed immersion of locally ringed spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on Z. Then there are canonical isomorphisms*

$$H^i(Z, \mathcal{F}) \to H^i(X, f_*\mathcal{F}) \qquad (i \geq 0).$$

Proof. We already know that f_* preserves flasqueness (previous exercise). We can also see that f_* is exact by its behavior on stalks:

$$(f_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in Z \\ 0 & x \notin Z. \end{cases}$$

Finally, we note that we have a canonical isomorphism at the level of H^0. If we then start with a flasque resolution of $\mathcal{F}$, we can either push forward (obtaining another flasque resolution) and then take global sections, or take global sections directly, to obtain the *same complexes*, and in particular the same cohomology. $\qquad\square$

This will very often allow us to carry out induction arguments on dimension, by setting up a short exact sequence of sheaves on X in which one member is the direct image of a sheaf on Z.

Lemma. *Let X be a noetherian topological space, and let $(\mathcal{F}_j)$ be a direct system of abelian sheaves. Then the natural functoriality maps*

$$\varinjlim H^{\cdot}(X, \mathcal{F}_j) \to H^{\cdot}(X, \varinjlim \mathcal{F}_j)$$

(coming from the maps $\mathcal{F}_j \to \varinjlim \mathcal{F}_j$) are isomorphisms.

Proof. Fix the index set for the direct system. Then both sides are cohomological functors on the category of direct systems of abelian sheaves for that index set, and the noetherian hypothesis forces the H^0 terms to match (earlier exercise), so it suffices to check that they are both effaceable. For that, it suffices to observe that there is a *functorial* way to embed any sheaf of abelian groups on X into a flasque sheaf; for instance, see Hartshorne exercise II.1.16(e). $\qquad\square$

3   The result

Before stating the theorem, I need to make one general observation. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules on a scheme X, with $\mathcal{F}$ quasicoherent and flat. I claim there is a natural homomorphism

$$H^0(X, \mathcal{F}) \otimes_{\mathcal{O}_X} H^r(X, \mathcal{G}) \to H^r(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

of $\Gamma(X, \mathcal{O}_X)$-modules for any nonnegative integer r. It comes from the facts that:

- both sides are cohomological functors in $\mathcal{G}$ (using the fact that $\mathcal{F}$ is flat on the right side, and the fact that $H^0(X, \mathcal{F})$ is flat over $\mathcal{O}(X)$ on the left side);

- the left side is effaceable and hence universal;

- there is a natural map for $r = 0$.

We can also compute this map using Čech complexes.

Theorem (Serre). *Let A be any ring, fix an integer $r \geq 1$, and put $X = \mathbb{P}_A^r$ and $S = A[x_0, \ldots, x_n]$.*

(a) *The natural map*

$$S \to \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$$

is an isomorphism of graded S-modules.

(b) *For $0 < i < r$ and $n \in \mathbb{Z}$, we have $H^i(X, \mathcal{O}_X(n)) = 0$.*

(c) *We have $H^r(X, \mathcal{O}_X(-r-1)) \cong A$.*

(d) *The natural A-bilinear map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$. (That is, each side is isomorphic to Hom of the other side into A; in particular, they have the same rank.)

(e) *For $i > r$, we have $H^i(X, \mathcal{O}_X(n)) = 0$.*

Proof of the theorem. It is enough to compute the cohomology of the sheaf $\mathcal{F} = \oplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$ and keep track of the grading by n. (This can be seen by applying the previous lemma, but this forces a noetherian hypothesis. But it is also clear without such a hypothesis, because the Čech cohomology computation we are about to do commutes with the direct sums.) For starters, recall that we checked part (a) earlier; see for instance Hartshorne Proposition II.5.13.

We use the obvious Čech resolution by the $D_+(x_i)$ for $i = 0, \ldots, r$. Since the complex vanishes above degree r, we immediately get (e). (This also follows from a general theorem of Grothendieck; see Hartshorne Theorem III.2.7.) To compute $H^r(X, \mathcal{F})$, we need the cokernel of

$$d^{r-1} : \prod_{k=0}^{r} S_{x_0 \cdots x_{k-1} x_{k+1} \cdots x_r} \to S_{x_0 \cdots x_r}.$$

View $S_{x_0 \cdots x_r} = A[x_0^{\pm}, \ldots, x_r^{\pm}]$ as the free A-module generated by the monomials $x_0^{e_0} \cdots x_r^{e_r}$ with $e_0, \ldots, e_r \in \mathbb{Z}$. The image under d^{r-1} of the k-th factor of the product is precisely the span of the monomials with $e_k \geq 0$. Hence $H^r(X, \mathcal{F})$ is the free A-module generated by the $x_0^{e_0} \cdots x_r^{e_r}$ with $e_0, \ldots, e_r < 0$, graded by degree. In particular, in degree $-r - 1$, we see exactly one monomial $x_0^{-1} \cdots x_r^{-1}$, proving (c).

To see (d), we must make the pairing explicit. First, note that there is nothing to check if $n < 0$, since then $H^0(X, \mathcal{O}_X(n)) = S_{-n} = 0$ obviously, and $H^0(X, \mathcal{O}_X(-n - r - 1)) = 0$ because there are no monomials of degree greater than $-r - 1$ with all exponents negative. So assume hereafter $n \geq 0$. If we identify $H^0(X, \mathcal{O}_X(n))$ with the A-span of the monomials in $x_0, \ldots, x_r$ (with nonnegative powers) of degree n, then the pairing with $H^r(X, \mathcal{F})$ (the A-span of the monomials of degree $-n - r - 1$ in $x_0^{-1}, \ldots, x_r^{-1}$) is to simply multiply together and throw away everything except the term $x_0^{-1} \cdots x_r^{-1}$. This implies (c).

It remains to prove (b), which we do by induction on r. The base case is $r = 1$, for which there is nothing to check because $0 < i < r$ is impossible. Before running the induction, we note that if we localize the Čech complex by inverting x_r, we get the corresponding Čech complex on the open set $D_+(x_r)$, which is affine. So the localized Čech complex must be acyclic since it computes the cohomology of a quasicoherent sheaf on an affine scheme. On the other hand, localizing in x_r is exact, so it commutes with taking cohomology; that is,

3

the localization $H^i(X, \mathcal{F})_{x_r} = 0$ for $i > 0$. In other words, every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

It thus suffices to show that for $0 < i < r$, multiplication by any power of x_r is injective on $H^i(X, \mathcal{F})$; it also suffices to check multiplication by x_r itself. Look at the exact sequence

$$0 \to S(-1) \overset{\times x_r}{\to} S \to S/(x_r) \to 0$$

of graded S-modules. Writing $H \cong \mathbb{P}_A^{r-1}$ for the hyperplane $x_r = 0$ and $j : H \to X$ for the inclusion, this sheafifies to

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \oplus_{n \in \mathbb{Z}}(j_* \mathcal{O}_H)(n) \to 0.$$

Let's take the long exact sequence in homology. In degree 0 we get back our original sequence, so the connecting homomorphism into $H^1(X, \mathcal{F}(-1))$ is zero. That (which holds even in the base case) plus the induction hypothesis, which implies that $H^i(X, \oplus_n(j_* \mathcal{O}_H)(n)) = H^i(H, \oplus_n \mathcal{O}_H(n)) = 0$ for $0 < i < r - 1$, gives us the bijectivity of multiplication by x_r on $H^i(X, \mathcal{F})$ for $i = 0, \ldots, r-2$ and injectivity for $i = r-1$. This is enough to get $H^i(X, \mathcal{F}) = 0$ for $0 < i < r$. $\square$

4   Finiteness of cohomology on projective schemes

Using the previous calculation, Serre was able to derive a powerful finiteness and vanishing theorem for cohomology on projective schemes. First, we need another result of Serre (Hartshorne II.5.17 except without the noetherian hypothesis).

Theorem. *Let A be a ring, let $X \to \mathbb{P}_A^r$ be a closed immersion for some $r \geq 1$, and let $\mathcal{O}_X(1)$ be the pullback of the twisting sheaf. Let $\mathcal{F}$ be a finitely generated quasicoherent sheaf on X. Then there exists an integer n_0 such that for all $n \geq n_0$, $\mathcal{F}(n)$ is generated by a finite number of global sections.*

Proof. By replacing $\mathcal{F}$ by its direct image, we reduce to the case $X = \mathbb{P}_A^r$ itself. For $i = 0, \ldots, r$, we have $\mathcal{F}|_{D_+(x_i)} = \tilde{M_i}$ for some finitely generated module M_i over $B_i = A[x_0/x_i, \ldots, x_r/x_i]$. For any $s \in M_i$, for $n \geq n_0$ for some n_0 depending on s, $x_i^n s$ is a section of $\mathcal{F}(n)$. For n_0 large enough, we can lift a set of generators of each M_i to sections of $\mathcal{F}(n)$ whenever $n \geq n_0$; this proves the claim. $\square$

Corollary. *With notation as in the previous theorem, we obtain a surjection $\oplus_{i=1}^m \mathcal{O}(n) \to \mathcal{F}$ for some $n \in \mathbb{Z}$.*

For this I need a noetherian hypothesis, but only until we define coherent sheaves.

Theorem (Serre)**.** *Let A be a noetherian ring, let $X \to \mathbb{P}_A^r$ be a closed immersion for some $r \geq 1$, and let $\mathcal{O}_X(1)$ be the pullback of the twisting sheaf. Let $\mathcal{F}$ be a finitely generated quasicoherent sheaf on X.*

(a) *The A-modules $H^i(X, \mathcal{F})$ are finitely generated for $i \geq 0$.*

(b) *There exists an integer n_0 (depending on $\mathcal{F}$) such that for each $i > 0$ and $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.*

Proof. We proceed by *descending* induction on i. For $i > r$, we have $H^i(X, \mathcal{F}) = 0$ because X admits a good over by at most $r + 1$ open affines.

By the previous corollary, we can write $\mathcal{F}$ as a quotient of some sheaf $\mathcal{E}$ which is a direct sum of twisting sheaves. Let $\mathcal{G}$ be the kernel:

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

Thanks to the noetherian hypothesis, we may conclude that $\mathcal{G}$ is also finitely generated. The long exact sequence in cohomology gives:

$$\cdots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{G}) \to \cdots$$

Given the claim for $i + 1$, we know that the right term is finitely generated as an A-module. The left term is a sum of things of the form $H^i(X, \mathcal{O}_X(n))$ for various $n \in \mathbb{Z}$, and we already computed those and saw that they were finitely generated as A-modules. Again since A is noetherian, we can conclude that the middle module is finitely generated. This proves (a).

To get (b), twist by n and then again take the long exact sequence in cohomology:

$$\cdots \to H^i(X, \mathcal{E}(n)) \to H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{G}(n)) \to \cdots$$

For n large, the right module vanishes by the induction hypothesis, while the left module vanishes by the explicit calculation, so the middle group vanishes. $\square$