18.726 Algebraic Geometry
Spring 2009

18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009)
Projective morphisms, part 1 (updated 3 Mar 08)

We now describe projective morphisms, starting over an affine base.

1 Proj of a graded ring

The construction of Proj of a graded ring was assigned as an exercise; let me now recall the result of that exercise.

Let $S = \oplus_{n=0}^{\infty} S_n$ be a *graded ring*, i.e., a ring such that each S_n is closed under addition, and $S_m S_n \subseteq S_{m+n}$. An element of S_n is said to be *homogeneous of degree* n; the elements of S_0 form a subring of S, and each S_n is an S_0-module. (One could also define a graded ring to allow negative degrees; on the few occasions where I'll need that construction, I'll call it a *graded ring with negative degrees.*) Let S^+ denote the ideal $\oplus_{n=1}^{\infty} S_n$.

Let $\operatorname{Proj} S$ be the set of all homogeneous prime ideals of S not containing S_+. For each positive integer n and each $f \in S_n$, we may view the localization S_f as a graded ring with negative degrees, by placing g/f^k in degree $m - kn$ whenever $g \in S_m$. We may then identify the set

$$D(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$$

with $\operatorname{Spec} S_{f,0}$, where $S_{f,0}$ is the degree zero subring of S_f. These glue to equip $\operatorname{Proj} S$ with the structure of a scheme (note that $D(f) \cap D(g) = D(fg)$). In the case $S = A[x_0, \dots, x_n]$ where each of $x_0, \dots, x_n$ is homogeneous of degree 1, this simply produces the projective space $\mathbb{P}^n_A$.

Any morphism $S \to T$ of graded rings induces a morphism $\operatorname{Proj} T \to \operatorname{Proj} S$ of schemes. For example, we say an ideal I of S is *homogeneous* if as abelian groups we have

$$I = \oplus_{n=0}^{\infty} (I \cap S_n).$$

In other words, if we split each element of I into homogeneous components, the components themselves belong to I. Then S/I may also be viewed as a graded ring, the projection $S \to S/I$ induces a morphism $\operatorname{Proj} S/I \to \operatorname{Proj} S$, and this morphism is a closed immersion (as we see immediately by checking on a $D(f)$).

Beware that the scheme $\operatorname{Proj} S$ does not by itself determine the graded ring S. For instance, omitting S_1 gives another graded ring with the same Proj. We'll come back to this point later.

More generally, if $M = \oplus_{n=-\infty}^{\infty}$ is a *graded* S*-module*, i.e., $S_m M_n \subseteq M_{m+n}$ for all m, n, we can convert M into a quasicoherent sheaf $\tilde{M}$ on $\operatorname{Proj} S$ by doing so on each $D(f)$ (using the degree-zero subset of M_f) and then glueing. For a converse, see below.

2 The sheaf $\mathcal{O}(1)$

For S a graded ring, n a nonnegative integer, and M a graded S-module, let $M(n)$ denote the shifted module

$$M(n)_i = M_{n+i}.$$

Let $\mathcal{O}_X(n)$ be the quasicoherent sheaf on $X = \operatorname{Proj} S$ defined by the graded module $S(n)$. In particular, $\mathcal{O}_X(0) = \mathcal{O}_X$. More generally, for any quasicoherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, put $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Lemma. *Suppose that S is generated by S_1 as an S_0-algebra. Then the sheaves $\mathcal{O}_X(n)$ on $\operatorname{Proj} S$ are locally free of rank 1, and $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is canonically isomorphic to $\mathcal{O}_X(m+n)$.*

Proof. See Hartshorne, Proposition II.5.12. □

Note: a quasicoherent sheaf $\mathcal{F}$ on a locally ringed space X which is locally free of rank 1 is also called an *invertible sheaf.* That is because there is a unique sheaf $\mathcal{F}^\vee$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$, the *dual* of X (exercise). In this case, the dual of $\mathcal{O}_X(n)$ is in fact $\mathcal{O}_X(-n)$.

This gives us an explanation for what $x_0, \dots, x_n$ are on the projective space $\operatorname{Proj} A[x_0, \dots, x_n]$: they are global sections not of the sheaf $\mathcal{O}_X$, but rather of the sheaf $\mathcal{O}_X(1)$.

Theorem 1. *Suppose that S is finitely generated by S_1 as an S_0-algebra. Then each quasicoherent sheaf on $\operatorname{Proj} S$ can be written as $\tilde{M}$ for a canonical choice of M.*

The finitely generated hypothesis is needed to ensure that $\operatorname{Proj} S$ is quasicompact; we will impose it pretty consistently hereafter.

Proof. Let $\mathcal{F}$ be a quasicoherent sheaf on M. Then the module we want is

$$\Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)),$$

where

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

For the rest of the proof, see Hartshorne, Proposition II.5.15. □

Beware that this does not imply that $S = \oplus_{n=0}^\infty \Gamma(X, \mathcal{O}_X(n))$ in general. For a stupid example, take $S = A[x]$, in which case the sheaves $\mathcal{O}_X(n)$ are all free and so $\Gamma(X, \mathcal{O}_X(n)) \neq 0$ even when $n < 0$. For less stupid examples, see Hartshorne exercise II.5.14. However, the following is true.

Lemma. *Let $n \geq 1$ be an integer. For $S = A[x_0, \dots, x_n]$ with the usual grading (by total degree), we have*

$$S = \oplus_{n=0}^\infty \Gamma(X, \mathcal{O}_X(n)).$$

Proof. Exercise, or see Hartshorne Proposition II.5.13. □

3 Closed subschemes of projective spaces

Proposition. *For $n \geq 1$, any closed immersion into $\mathbb{P}^n_A$ is defined by some homogeneous ideal of $A[x_0, \ldots, x_n]$.*

Proof. In fact, there is a canonical way to pick out the ideal. Let $\mathcal{I}$ be the ideal sheaf defining the closed immersion; then $\Gamma_*(\mathcal{I})$ is an ideal of $\Gamma_*(\mathcal{O}_X)$, but we already identified the latter with $S = A[x_0, \ldots, x_n]$. (This identification uses the fact that S is finitely generated by S_1 as an S_0-algebra, in order to invoke the previous theorem. In fact, it is part of the proof of that theorem; see Hartshorne Proposition II.5.13.) □

In general, there may be multiple homogeneous ideals defining the same closed subscheme of $\mathbb{P}^n_A$. If we start with an ideal I, pass to the closed subscheme, then use the previous proposition to get back, we get the *saturation* of I, namely, the set of all elements $f \in A[x_0, \ldots, x_n]$ such that $x_0^j f, \ldots, x_n^j f \in I$ for some nonnegative integer j. We thus obtain a one-to-one correspondence between closed subschemes of $\mathbb{P}^n_A$ and *saturated* (equal to their saturation) homogeneous ideals.

Corollary. *For $n \geq 1$, let I be a homogeneous ideal of $S = A[x_0, \ldots, x_n]$. The following conditions are equivalent.*

(a) The subscheme of $\mathbb{P}^n_A$ defined by I is empty.

(b) The saturation of I equals S^+.

(c) For some n_0, we have $S_n \subseteq I$ for all $n \geq n_0$.

Proof. We just proved the equivalence of (a) and (b). It is clear that (c) implies (b). Let us check that (b) implies (c). Given (b), each $f \in \{x_0, \ldots, x_n\}$ has the property that $x_0^j f, \ldots, x_n^j f \in I$ for some j. In particular, we have $x_0^j, \ldots, x_n^j \in I$ for some j. This in turn implies $S_{(n+1)(j-1)+1} \subseteq I$ since each monomial of degree $(n+1)(j-1)+1$ is divisible by one of $x_0^j, \ldots, x_n^j$ (pigeonhole principle!). □

4 Projective implies proper

We are now ready to complete the proof that $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper. Recall that the missing step was to show that f is universally closed, i.e., for any scheme X, the map $\mathbb{P}^n_X \to X$ is closed. It is enough to do this in case $X = \operatorname{Spec} A$ is affine. Moreover, we may assume $n \geq 1$, as the case $n = 0$ is stupid (because f is an isomorphism).

Let Z be a closed subset of $\mathbb{P}^n_X$, suppose $z \in X$ is not in the image of Z, and put $k = \kappa(z)$. We must exhibit an open neighborhood U of x in X such that $Z \cap \mathbb{P}^n_U = \emptyset$. Let $I = \oplus_{n=0}^{\infty} I_n$ be the saturated homogeneous ideal in $S = A[x_0, \ldots, x_n]$ defining Z. Then $I \otimes_A k$ defines the empty subscheme of $\operatorname{Proj} k[x_0, \ldots, x_n]$, but may not be saturated. Nonetheless, for some m, we have that $I_n \otimes_A k = S_n \otimes_A k$, and so $(S_n/I_n) \otimes_A k = 0$.

Since S_n/I_n is a finitely generated A-module, by Nakayama's lemma, $(S_n/I_n) \otimes_A A_{\mathfrak{p}} = 0$ for $\mathfrak{p}$ the prime ideal of A defining z. Again since S_n/I_n is finitely generated, we have $(S_n/I_n) \otimes_A A_g = 0$ for some $g \in A \setminus \mathfrak{p}$. Then $z \in D(g)$ and $D(g)$ is disjoint from the image of Z, proving the claim.

5 What is a projective morphism?

Several authors (Hartshorne, Eisenbud-Harris) define a morphism $f : Y \to X$ to be *projective* if it is the composition of a closed immersion $Y \to \mathbb{P}^n_X$ with the projection $\mathbb{P}^n_X$ for some nonnegative integer n. This definition is evidently stable under base change, but it is *not local on the base*! Better to say that such a morphism is *globally projective*, and to say that f is *locally projective* if each $x \in X$ admits an open neighborhood U such that $f : Y \times_X U \to U$ is globally projective.

This is not such a serious distinction in practice, as globally projective equals locally projective if X is "not too large". For instance, this occurs if X is itself *globally quasiprojective* over an affine scheme. (A morphism is *globally quasiprojective* if it factors as an open immersion followed by a globally projective morphism. Again, this is stable under base change but not local on the base; the version where we force locality on the base is a *quasiprojective* morphism.)

The definition of *projective* given in EGA is in fact somewhere between locally and globally projective. More on that later. (Warning: Eisenbud-Harris claim that locally projective and projective are the same. They aren't, but counterexamples are rather pathological.)