18.725: EXERCISE SET 2

DUE TUESDAY SEPTEMBER 23

(1) Let $\Sigma = k - \{0\}$. Show that Σ is naturally an affine algebraic set, and that the natural map $\Sigma \hookrightarrow k$ is a morphism.

(2) Let $f : \Sigma_1 \longrightarrow \Sigma_2$ be a morphism of affine algebraic sets, and say $p \in \Sigma_2$ is a point. Show that $f^{-1}(p)$ has a natural structure of an affine algebraic set so that the inclusion $f^{-1}(p) \hookrightarrow \Sigma_1$ is a morphism.

(3) Show that $\Sigma = \{(x, y, z) \in k^3 | x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible algebraic subset of k^3 and find $I(\Sigma)$.

(4) Let $f : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $f(t) = (t^2, t^3)$. Is f an isomorphism? Prove or disprove.

(5) Show that a basis for the Zariski topology on k^n is given by the sets $D(f) := k^n - V(f)$, for $f \in k[X_1, \ldots, X_n]$. Deduce that for any algebraic set $\Sigma \subset k^n$, a basis for the Zariski topology on Σ is given by the sets $\Sigma - V(f)$ for $f \in \Gamma(\Sigma)$. We also denote these subsets by $D(f)$.

(6) Let $\Sigma_1 \subset \mathbb{P}^2$ and $\Sigma_2 = \mathbb{P}^1$ be as in the example on page 22, and let $f : \Sigma_1 \to \Sigma_2$ be the map defined there. Let $U_i \subset \mathbb{P}^2$ be the standard covering $(i = 0, 1, 2)$ of $\mathbb{P}^2$, and $V_j \subset \mathbb{P}^1$ $(j = 0, 1)$ the standard covering of $\mathbb{P}^1$. Show that f induces morphisms of affine algebraic sets

$$\Sigma_1 \cap U_0 \to V_1, \quad \Sigma_1 \cap U_2 \to V_0.$$

What about $\Sigma_1 \cap U_1$?

(7) Show that as a topological space $U = \mathbb{P}^n - \{(0 : X_1 : \cdots : X_n)\}$ is homeomorphic to k^n with the Zariski topology. Here the topology on U is that induced by the Zariski topology on $\mathbb{P}^n$.

(8) Is the Zariski topology on an affine algebraic set $\Sigma \subset k^n$ Hausdorff? What about Kolmogoroff (i.e. given any two distinct points there exists an open containing one but not the other)? Prove or give counterexamples.

(9) Show that any open set V of an affine algebraic set Σ is quasi–compact (i.e. every open cover admits a finite subcover).

(10) Is the image of a morphism $f : \Sigma_1 \to \Sigma_2$ between algebraic sets again an algebraic set? Prove or disprove.

Correction: In problem 6, the reference should be to the example on the bottom of page 15 (in the LNM 1358 version). Also, in the displayed equation V_0 and V_1 should beinterchanged.

Date: September 16, 2003.