10/15/04.

<u>Mordell's Theorem</u>: Let C be a non-singular cubic, given by
$y^2 = x^3 + ax^2 + bx$.
where $a, b \in \mathbb{Z}$. Then the group of rational points $\Gamma = C(\mathbb{Q})$ is finitely generated.

<u>Lemma 1</u>: For any real number M, the set
$\{ P \in \Gamma \mid h(P) \leq M \}$ is finite.

<u>Lemma 2</u>: Let P_0 be fixed rational point on C. Then $\exists k_0$ depending on P_0, a, b, c s.t. $h(P + P_0) \leq 2h(P) + k_0$.
all $P \in \Gamma$.

<u>Lemma 3</u>: There is a constant k depending on a, b, c s.t.
$h(2P) \geq 4h(P) - k$.

and Descent Theorem: Lemmas 1–4 imply Mordell's Theorem.

<u>Lemma 4</u>: $\langle \Gamma : 2\Gamma \rangle$ is finite.

Generally, we could work in the field obtained by adjoining a root of $f(x)$ to $\mathbb{Q}$.
We will prove for curves s.t. $f(x_0) = 0$ for some $x_0 \in \mathbb{Q}$.
Since f is ~~monic~~, $x_0 \in \mathbb{Z}$.

Then $T = (0,0)$ is of order 2
C is nonsingular $\Rightarrow$ $D = b^2(a^2 - 4b) \neq 0 \Rightarrow b \neq 0$,
$a^2 \neq 4b$.

<u>Duplication map</u> : $P \longmapsto 2P$. is of degree 4

$$\begin{tikzcd} C \arrow[loop left, "\text{duplication}", "\deg 4"'] \arrow[r, bend left, "\varphi"] & \bar{C} \arrow[l, bend left, "\psi"] \end{tikzcd}$$

Define $\bar{C}$ by $y^2 = x^3 + \bar{a}x^2 + \bar{b}x$, where
$\bar{a} = -2a$, $\bar{b} = a^2 - 4b$.

$\bar{\bar{C}}$ by $y^2 = x^3 + \bar{\bar{a}}x^2 + \bar{\bar{b}}x$, where

$\bar{\bar{a}} = -2\bar{a} = 4a$

$\bar{\bar{b}} = \bar{a}^2 - 4\bar{b} = 16b$.

$\bar{\bar{C}}$: $y^2 = x^3 + 4ax^2 + 16bx$.

Substitute $y \longmapsto 8y$

$x \longmapsto 4x$

$$(8y)^2 = (4x)^3 + 4a(4x)^2 + 16b(4x)$$

$$\cancel{64}y^2 = \cancel{64}x^3 + \cancel{64}a x^2 + \cancel{64}bx$$

Let $\bar{\Gamma}$, $\bar{\bar{\Gamma}}$ be the groups of rational points in $\bar{C}$, $\bar{\bar{C}}$.

Define $\varphi : \Gamma \longrightarrow \bar{\Gamma}$ a homomorphism $\boxed{\Gamma \cong \bar{\bar{\Gamma}}}$

$\psi : \bar{\Gamma} \longrightarrow \bar{\bar{\Gamma}}$

$\psi \circ \varphi(P) = 2P$.

$\varphi(x, y) = (\bar{x}, \bar{y})$ where $\bar{x} = x + a + \frac{b}{x} = \frac{y^2}{x^2}$,
$\bar{y} = y\left(\frac{x^2 - b}{x^2}\right)$

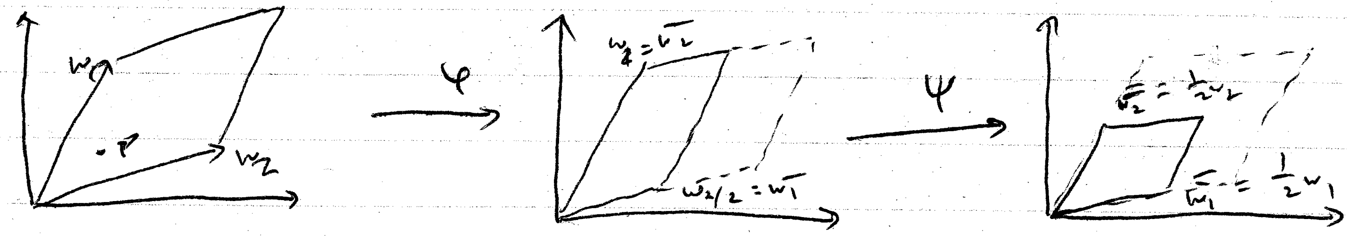
$$
\begin{aligned}
\bar{x}^3 + \bar{a}\bar{x}^2 + \bar{b}\bar{x} &= \bar{x}\left(\bar{x}^2 - 2a\bar{x} + (a^2 - 4b)\right) \\
&= \left(\frac{y^2}{x^2}\right)\left(\frac{y^4}{x^4} - 2a\frac{y^2}{x^2} + a^2 - 4b\right) \\
&= \left(\frac{y^2}{x^2}\right)\left(\frac{y^4 - 2ay^2x^2 + a^2x^4 - 4bx^4}{x^4}\right) \\
&= \left(\frac{y^2}{x^2}\right)\left(\frac{(y^2 - ax^2)^2 - 4bx^4}{x^4}\right) \\
& \left(\frac{y^2}{x^2}\right)\left(\frac{(x^3 + bx)^2 - 4bx^4}{x^4}\right) \\
&= \left(\frac{y^2}{x^6}\right)\left(x^6 - 2bx^4 + b^2x^2\right) \\
&= \left(\frac{y^2}{x^6}\right)\left(x^2(x^2 - b)^2\right) \\
&= \left(\frac{y(x^2 - b)}{x^2}\right)^2 = \bar{y}^2
\end{aligned}
$$

So $(\bar{x}, \bar{y}) \in \bar{\Gamma}$.

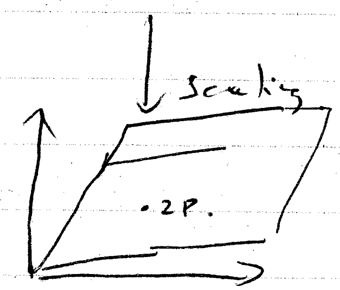
Also Let $\varphi(T) = \bar{O}$
$\varphi(O) = \bar{O}$

Recall $\wp(u + w_1) = \wp(u)$, $\wp(u + w_2) = \wp(u)$ $w_1, w_2 \in \mathbb{C}$
and then

Def $P: \mathbb{C} \longrightarrow \Gamma$ given by $P(u) = (\wp(u), \wp'(u))$



$P(u_1+u_2) = P(u_1)+P(u_2)$



Algebraic: C is abelian, $\{O, \underset{(0,0)}{T}\}$ is a subgroup of C.

So in some sense $\overline{C} \cong C/\{O,T\}$.

$\overline{\Gamma} \cong \Gamma/\{O,(T)\}$

Proposition. Let $C, \overline{C}$ be given by

$C: y^2 = x^3+ax^2+bx \qquad \overline{C}: y^2 = x^3+\overline{a}x^2+\overline{b}x$

$(\overline{a} = -2a, \ \overline{b} = a^2-4b)$.

Let $T=(0,0) \in C$.

(a) $$\varphi(P) = \begin{cases} \left(y^2/x^2, \dfrac{y(x^2-b)}{x^2}\right) & \text{if } P(x,y) \neq O, T \\ \overline{O} & \text{if } P \in \{O, T\} \end{cases}$$

is a homomorphism.

(b). Applying this to $\overline{C}$ gives a map $\overline{\varphi}$ from $\overline{C}$ to $\overline{\overline{C}}$. $\overline{\overline{C}} \cong C$ via the map $(x,y) \mapsto (x/4, y/8)$

There is a homomorphism $\psi: \overline{C} \longrightarrow C$.

$$\psi(P) = \begin{cases} \left(\dfrac{\bar{y}^2}{4\bar{x}^2}, \dfrac{\bar{y}(\bar{x}^2 - \bar{b})}{8\bar{x}^2} \right) & \text{if } \bar{P} = (\bar{x}, \bar{y}) \notin \{\bar{O}, \bar{T}\} \\ O & \text{if } \bar{P} \in \{\bar{O}, \bar{T}\}. \end{cases}$$

and $\psi \circ \varphi(P) = 2P$.