Convergence of Random Variables
Probability Inequalities

MIT 18.655

Dr. Kempthorne

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Outline

1. Convergence, Probability Inequalities
   - Convergence of Random Variables
   - Probability Inequalities

Convergence of Random Variables/Vectors

Framework

- $\mathbf{Z}_n = (Z_{n,1}, Z_{n,2}, \ldots, Z_{n,d})^T$, a sequence of random vectors.
- $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_d)^T$, a random vector (e.g., sequence limit)

Definitions/Terminology/Theorems

(B.7.1) "Convergence in Probability":

$\{\mathbf{Z}_n\}$ **converges in probability** to $\mathbf{Z}$.

$|\mathbf{Z}_n - \mathbf{Z}| \xrightarrow{P} 0$

$\mathbf{Z}_n \xrightarrow{P} \mathbf{Z}$

Definition: For every $\epsilon > 0$: $\lim_{n\to\infty} P(|Z_n - Z| > \epsilon) \to 0$.

NOTE(!): Convergence in Probability **REQUIRES** joint distribution of $\mathbf{Z}_n$ and $\mathbf{Z}$.

Convergence of Random Variables/Vectors

Definitions/Terminology (continued)

(B.7.2) Convergence in Law / Convergence in Distribution:

$\{\mathbf{Z}_n\}$ **converges in law** to $\mathbf{Z}$.

$\mathbf{Z}_n \xrightarrow{\mathcal{L}} \mathbf{Z}$

$\mathcal{L}(\mathbf{Z}_n) \to \mathcal{L}(\mathbf{Z})$

Definition: for every $\mathbf{t} \in R^d$, where the distribution function $F_{\mathbf{Z}}$ of $\mathbf{Z}$ is continuous:

$$\lim_{n\to\infty} F_{\mathbf{Z}_n}(\mathbf{t}) = F_{\mathbf{Z}}(\mathbf{t}).$$

NOTE(!): Convergence in Law/Distribution does **NOT** use joint distribution of $\mathbf{Z}_n$ and $\mathbf{Z}$.

(A.14.4) If $Z = z_0$, a constant, convergence in law/distribution implies convergence in probability:

$$Z_n \xrightarrow{\mathcal{L}} z_0 \Longrightarrow Z_n \xrightarrow{P} z_0.$$

Convergence of Random Variables

(A.14.6) If $Z_n \xrightarrow{P} z_0$, and g is continuous at z_0, then

$$g(Z_n) \xrightarrow{P} g(z_0).$$

(A.14.8) If $Z_n \xrightarrow{\mathcal{L}} Z$, and g is continuous, then

$$g(Z_n) \xrightarrow{\mathcal{L}} g(Z).$$

Theorem (A.14.9) If $Z_n \xrightarrow{\mathcal{L}} Z$, and $U_n \xrightarrow{P} u_0$, a constant, then

(a). $Z_n + U_n \xrightarrow{\mathcal{L}} Z + u_0$,

(b). $U_n Z_n \xrightarrow{\mathcal{L}} u_0 Z$.

Convergence of Random Variables

Corollary (A.14.17) Suppose

- $\{a_n\}$: $\lim_{n\to\infty} a_n = \infty$.
- b: $-\infty < b < \infty$, a fixed number.
- $a_n(Z_n - b) \xrightarrow{\mathcal{L}} Z$.
- $g(\cdot)$: a function of a real variable whose derivative, g', exists and is continuous at b.

Then

$$a_n[g(Z_n) - g(b)] \xrightarrow{\mathcal{L}} g'(b)Z.$$

Convergence of Random Variables/Vectors

Theorem B.7.2 Slutsky's Theorem. Suppose $\mathbf{Z}_n^T = (\mathbf{U}_n{}^T, \mathbf{V}_n{}^T)$ where $\mathbf{Z}_n$ is a d-vector, $\mathbf{U}_n$ is a b-vector, $\mathbf{V}_n$ is a c-vector

$(d = b + c)$

- $\mathbf{U}_n \xrightarrow{\mathcal{L}} \mathbf{U}$
- $\mathbf{V}_n \xrightarrow{\mathcal{L}} \mathbf{v}$, a constant vector
- $\mathbf{g} : R^d \to R^b$ is continuous

Then

$$\mathbf{g}(\mathbf{U}_n{}^T, \mathbf{V}_n{}^T) \xrightarrow{\mathcal{L}} \mathbf{g}(\mathbf{U}^T, \mathbf{V}^T).$$

Examples:

(a). $d = 2$, $b = c = 1$,

$g(u, v) = \alpha u + \beta v$, or

$g(u, v) = u/v$

Slutsky's Theorem

Examples (continued):

(b). $\mathbf{V}_n\mathbf{U}_n + \mathbf{W}_n \xrightarrow{\mathcal{L}} \mathbf{v}\mathbf{U} + \mathbf{w}$, where
$\{\mathbf{V}_n\}$ matrix r.v.'s $(r \times d)$
$\mathbf{V}_n \xrightarrow{P} \mathbf{v}$ (constant matrix)
$\mathbf{U}_n \xrightarrow{\mathcal{L}} \mathbf{U}$
$\mathbf{W}_n \xrightarrow{P} \mathbf{w}$ (constant vector)
(all dimensions conformal)

Theorem B.7.4 If

- $\mathbf{U}_n \xrightarrow{P} \mathbf{U}$
- $g(\cdot)$: bounded and $P[\mathbf{U} \in A_g] = 1$,

Then: $Eg(\mathbf{U}_n) \to Eg(\mathbf{U})$.

Dominated Convergence Theorem

Theorem B.7.5 Dominated Convergence Theorem. If $\{W_n\}$, W and V are random variables with

- $W_n \xrightarrow{P} W$
- $P(|W_n| < |V|) = 1$, for all n
- $E[|V|] < \infty$

Then: $E[W_n] \to E[W]$.

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Inequalities

(a). Chebychev's Inequality: If X is any random variable, then

$$P[|X| \geq a] \leq \frac{E[X^2]}{a^2}.$$

(b). Markov's Inequality: If X is any random variable, then

$$P[|X| \geq a] \leq \frac{E[|X|]}{a}.$$

(c). Generalization: If X is any random variable, and $g(\cdot)$ is non-negative and non-decreasing on range of X:

$$P[X \geq a] \leq \frac{E[g(X)]}{g(a)}.$$

(d). Bernstein's Inequality: $g(t) = e^{st}$: If X is any random variable,

$$P[X \geq a] \leq \frac{E[e^{sX}]}{e^{sa}}.$$

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