18.600: Lecture 30

Central limit theorem

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Outline

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Proving the central limit theorem

Recall: DeMoivre-Laplace limit theorem

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- **Central limit theorem:** Yes, if they have finite variance.

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- Characteristic functions are well defined at all t for all random variables X.

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- Recall: the weak law of large numbers can be rephrased as the statement that $A_n = \frac{X_1+X_2+\ldots+X_n}{n}$ converges in law to μ (i.e., to the random variable that is equal to μ with probability one) as $n \to \infty$.

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- The central limit theorem can be rephrased as the statement that $B_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sigma\sqrt{n}}$ converges in law to a standard normal random variable as $n \to \infty$.

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- So g is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = 1$. Taylor expansion: $g(t) = t^2/2 + o(t^2)$ for t near zero.
- Now B_n is $\frac{1}{\sqrt{n}}$ times the sum of n independent copies of Y.
- So $M_{B_n}(t) = \left(M_Y(t/\sqrt{n})\right)^n = e^{ng(\frac{t}{\sqrt{n}})}$.
- But $e^{ng(\frac{t}{\sqrt{n}})} \approx e^{n(\frac{t}{\sqrt{n}})^2/2} = e^{t^2/2}$, in sense that LHS tends to $e^{t^2/2}$ as n tends to infinity.

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- But the proof can be repeated almost verbatim using characteristic functions instead of moment generating functions.
- Then it applies for any X with finite variance.

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- But $e^{ng(\frac{t}{\sqrt{n}})} \approx e^{-n(\frac{t}{\sqrt{n}})^2/2} = e^{-t^2/2}$, in sense that LHS tends to $e^{-t^2/2}$ as n tends to infinity.

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- *Kind of* true for homogenous population, ignoring outliers.