18.600 Midterm 2, Fall 2019 Solutions

1. (20 points)

(a) Melissa is applying to 20 different out-of-state medical schools. Because of her excellent GPA/MCAT/essays, her chance of being accepted to each school is 1/20, and the decisions at the 20 schools are independent of each other. Using a Poisson approximation, estimate the probability that Melissa will be accepted to at least two of these schools. **ANSWER:** Number X of acceptances is roughly Poisson with parameter $\lambda = 20 \cdot \frac{1}{20} = 1$. Thus $P(X \geq 2) = 1 - P(X = 1) - P(X = 0) \approx 1 - e^{-\lambda}\lambda^1/1! - e^{-\lambda}\lambda^0/0! = 1 - 2/e \approx .26424$. **Remark:** If we compute the exact value using a binomial distribution, we get $P(X \geq 2) \approx .26416$, so the approximation is quite good.

(b) Jill is applying to 25 different out-of-state medical schools and has a 1/5 chance (independently) of being invited for an interview at each school. Let X be the number of medical schools at which she is invited to interview. Compute $E[X]$ and $\text{Var}[X]$. **ANSWER:** The number of interviews is binomial with parameter $n = 25$ and $p = 1/5$. So $E[X] = np = 5$ and $\text{Var}[X] = np(1 - p) = 4$.

(c) Using a normal approximation, roughly approximate the probability that Jill is invited to interview at fewer than 2.5 schools. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** Since the standard deviation of X is 2, the value 2.5 is 5/4 standard deviations below the mean. Hence the probability is approximately $\Phi(-5/4) \approx .10565$. **Remark:** The true probability is .098 which is pretty close.

2. (20 points) A room has four lightbulbs, each of which will burn out at a random time. Let X_1, X_2, X_3, X_4 be the burnout times, and assume they are independent exponential random variables with parameter $\lambda = 1$. Write

1. $X = X_1 + X_2 + X_3 + X_4$.
2. $Y = \min\{X_1, X_2, X_3, X_4\}$, i.e., Y is time when first bulb burns out.
3. $Z = \max\{X_1, X_2, X_3, X_4\}$, i.e., Z is time when last bulb burns out.

Compute the following:

(a) The probability density function f_X. **ANSWER:** This is a Gamma distribution with parameters $\lambda = 1$ and $n = 4$. So $f_X(x) = x^3 e^{-x}/3!$ for $x \in [0, \infty)$.

(b) The probability density function f_Y. **ANSWER:** The minimum of four exponentials of parameter 1 is exponential with parameter 4. Hence $f_Y(x) = 4e^{-4x}$ for $x \in [0, \infty)$.

(c) The expectation $E[Z]$. **ANSWER:** This is basically the radioactive decay problem from lecture. Answer is $1/4 + 1/3 + 1/2 + 1$.

(d) The covariance $\text{Cov}(Y, Z)$. (Hint: use memoryless property.) **ANSWER:** The memoryless property implies that Y and $Z - Y$ are independent and hence $\text{Cov}(Y, Z) = \text{Cov}(Y, Y + (Z - Y)) = \text{Cov}(Y, Y) = \text{Var}(Y)$. Since Y is exponential with parameter $\lambda = 4$ its variance is $1/\lambda^2 = 1/16$.

3. (20 points) Five applicants are applying for a job, and an interviewer gives each applicant a score between 0 and 1. Call these scores $X_1, X_2, \ldots, X_5$ and assume that they are i.i.d. uniform random variables on $[0, 1]$. The top applicant has score $Y = \max\{X_1, X_2, \ldots, X_5\}$, and the second to the top has score Z, which we define to be the *second* largest of the X_i. Compute the following:

(a) The cumulative distribution function $F_Y(r)$ for $r \in [0, 1]$. **ANSWER:** $P(Y \le r) = P(\max\{X_1, X_2, \ldots, X_5\}) \le r) = P(X_1 \le r, X_2 \le r, \ldots) = P(X_1 \le r)^5 = r^5$.

(b) The density function f_Y. **ANSWER:** $f_Y(r) = F_Y'(r) = 5r^4$ for $r \in [0, 1]$ (and zero if $r \notin [0, 1]$).

(c) The density function f_Z and the value $E[Z]$. **NOTE:** If you remember what this means, you may use the fact that a Beta (a, b) random variable has expectation $a/(a + b)$ and density $x^{a-1}(1 - x)^{b-1}/B(a, b)$, where $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$. **ANSWER:** The ordering of candidates is independent of the set of scores obtained by the candidates. This means that the density of Z is the same that of a uniform random variable conditioned on three people being smaller, one being larger. This is a Beta (a, b) random variable with $a - 1 = 3$ and $b - 1 = 1$. So it comes to $x^3(1 - x)/B(4, 2) = 20x^3(1 - x)$ and $E[X] = 4/(4 + 2) = 2/3$.

(d) The probability $P(X_2 > 2X_1)$ (i.e., probability second candidate's score is more than than double first candidate's score). **ANSWER:** Note that joint density $f_{X_1,X_2}(x, y)$ is 1 on the unit square $[0, 1]^2$ and zero elsewhere. Therefore the probability is the area of the subset of $[0, 1]^2$ where $y > 2x$, which comes to $1/4$. So the answer is $1/4$.

4. (15 points) Let X and Y be independent random variables with density function given by $\frac{1}{\pi(1+x^2)}$.

(a) Compute $P(X < 1)$. **ANSWER:** X is a Cauchy random variable, so the answer is $3/4$ by our spinning flashlight story. Recall that in that story, we draw a line from $(0, 1)$ with a uniformly chosen angle and its intersection with $\mathbb{R}$ is a Cauchy random variable. The angle range corresponding to $(-\infty, 1)$ is $3/4$ of the total range, so the answer is $3/4$.

(b) Compute the probability density function for the random variable $Z = (X - Y)/2$. **ANSWER:** If Y is Cauchy then $-Y$ is also Cauchy. The average of two independent Cauchy random variables it itself Cauchy, so the answer is $\frac{1}{\pi(1+x^2)}$.

(c) Compute $E[e^{-X^2-Y^2}]$. You can leave your answer as a double integral—no need to evaluate it explicitly. **ANSWER:** $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi(1+x^2)}\frac{1}{\pi(1+y^2)}e^{-x^2-y^2}dxdy$

5. (10 points) Let $X_1, X_2, X_3, \ldots, X_{10}$ be the outcomes of independent standard die rolls—so each takes one of the values in $\{1, 2, 3, 4, 5, 6\}$, each with equal probability. Write $S = X_1 + X_2 + \ldots + X_{10}$. Compute the following:

(a) The moment generating function $M_{X_1}(t)$. **ANSWER:** $M_{X_1}(t) = E[e^{tX_1}] = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$.

(b) The moment generating function $M_S(t)$.**ANSWER:** The moment generating function of a sum of independent random variables is the product of the moment generating functions of the individual random variables. Hence $M_S(t) = \left(\frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})\right)^{10}$.

6. (15 points) Let X and Y be be random variables with joint density function $f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$. Write $Z = X + Y$.

(a) Compute $E[XY]$. **ANSWER:** X and Y are independent normal random variables, each with mean zero and variance one. Since they are independent we have $E[XY] = E[X]E[Y] = 0$. Alternatively, write $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy$. Then there are various ways to argue by symmetry that this must be zero.

(b) Compute the conditional expectation $E[Y|Z]$. That is, express the random variable $E[Y|Z]$ in terms of Z. **ANSWER:** We have $Z = E[Z|Z] = E[X|Z] + E[Y|Z]$. Since $E[X|Z]$ and $E[Y|Z]$ are the same by symmetry, the answer must be $Z/2$.

(c) Compute the probability $P(X^2 + Y^2 \leq 4)$. **ANSWER:** This can be computed using polar coordinates. The integral becomes
$\int_0^2\int_0^{2\pi}\frac{1}{2\pi}e^{-r^2/2}rd\theta dr = \int_0^2 e^{-r^2/2}rdr = -e^{-r^2/2}|_0^2 = -e^{-2} - (-1) = 1 - e^{-2} \approx .86466$