18.600 Midterm 2, Spring 2018 Solutions

1. (20 points) Suppose that X is a random variable with probability density function given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x/2 & 0 < x \leq 2 \\ 0 & x > 2 \end{cases}.$$

Suppose that Y is an independent random variable with the same probability density function. Write $Z = X^2 + Y^2$.

(a) Compute the joint density function $f_{X,Y}(x,y)$. **ANSWER:** $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ which is $xy/4$ when $0 < x \leq 2$ and $0 < y \leq 2$, and 0 otherwise.

(b) Compute $E[Z]$. (You should be able to get an explicit number.) **ANSWER:** $E[X^2 + Y^2] = E[X^2] + E[Y^2] = 2E[X^2] = 2\int_0^2 x^2 x/2 dx = 2x^4/8 \Big|_0^2 = 4$

(c) Compute $P(\max\{X,Y\} \leq 1)$. **ANSWER:** $P(\max\{X,Y\} \leq 1) = P(X \leq 1, Y \leq 1) = P(X \leq 1)^2 = \left(\int_0^1 x/2 dx\right)^2 = (1/4)^2 = 1/16$

2. (10 points) In a certain population, there are $n = 110000$ healthy people, each of whom has a $p = .01$ chance (independently of everyone else) of developing a certain disease during the course of a given decade. Let X be the number of people who develop the disease.

(a) Compute $E[X]$ and $\text{Var}[X]$. **ANSWER:** $E[X] = np = 1100$ and $\text{Var}[X] = npq = 1100 \cdot .99 = 1089$.

(b) Use a normal random variable to estimate $P(1100 < X < 1133)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** $SD(X) = \sqrt{\text{Var}[X]} = 33$ so this is probability X is between 0 and 1 standard deviations above mean, which is approximately $\Phi(1) - \Phi(0)$ by de Moivre Laplace.

3. (20 points) Suppose that X_1, X_2 and X_3 are independent random variables, each of which has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

Write $X = X_1 + X_2 + X_3$. Write $A = \min\{X_1, X_2, X_3\}$. Write $B = \max\{X_1, X_2, X_3\}$.

(a) Give a probability density function for X. **ANSWER:** This is Gamma distribution with parameters $n = 3$ and $\lambda = 1$, so $f_X(x) = x^2 e^{-x}/2!$ for $x \geq 0$

(b) Give a probability density function for A. **ANSWER:** Minimum of three rate one exponentials is exponential with rate three, so $f_A(x) = 3e^{-3x}$ for $x \geq 0$.

(c) Compute $E[B]$ and $\text{Var}[B]$. **ANSWER:** This is the "radioactive decay" problem. $E[B] = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$ and $\text{Var}[B] = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$.

4. (10 points) Suppose that X, Y, and Z are independent random variables, each of which has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$. Write $V = 3X$ and $W = X + Y + Z$.

(a) Compute the probability density function for V. **ANSWER:** $f_V(x) = \frac{1}{3} f_X(x/3) = \frac{1}{3\pi\left(1+(x/3)^2\right)}$.

(b) Compute the probability density function for W. **ANSWER:** W has the same law as V (by amazing property of Cauchy random variables) so $f_W(x) = f_V(x)$.

5.(10 points) Let X and Y be independent standard normal random variables (so each has mean zero and variance one).

(a) Compute $P(X^2 + Y^2 \leq 1)$. Give an explicit value. **ANSWER:** Since X and Y are independent we have $f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{-x^2-y^2} dxdy$. Switching to polar coordinates, this is $\int_0^1 \int_0^{2\pi} \frac{1}{2\pi} e^{-x^2/2-y^2/2} d\theta dr = \int_0^1 re^{-r^2} = -e^{-r^2/2}\Big|_0^1 = 1 - e^{-1/2}$.

(b) Compute $P(\max\{|X|, |Y|\} \leq 1)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** By independence of X and Y we have $P(\max\{|X|, |Y|\} \leq 1) = P(|X| \leq 1) P(|Y| \leq 1) = \left(\Phi(1) - \Phi(-1)\right)^2$.

6. (20 points) Let X and Y be independent uniform random variables on $[0,1]$ and write $Z = X + Y$.

(a) Compute the conditional expectation $E[X|Z]$. (That is, express the random variable $E[X|Z]$ as a function of the random variable Z.) **ANSWER:** Note that $E[X+Y|Z] = E[Z|Z] = Z$. By additivity of conditional expectation and symmetry $E[X+Y|Z] = E[X|Z] + E[Y|Z] = 2E[X|Z]$. So $E[X|Z] = Z/2$.

(b) Compute the conditional expectation $E[Z|X]$. (That is, express the random variable $E[Z|X]$ as a function of the random variable X.) **ANSWER:** $E[X+Y|X] = E[X|X] + E[Y|X] = X + E[Y|X]$. X and Y are independent so $E[Y|X] = E[Y] = 1/2$. Answer is $X + 1/2$.

(c) Compute the conditional variance $\mathrm{Var}[Z|Y]$. (That is, express the random variable $\mathrm{Var}[Z|Y]$ as a function of the random variable Y.) **ANSWER:** Given Y, the conditional law of Z is uniform on $[Y, Y+1]$, and thus the conditional variance is $1/12$ (regardless of the Y value) so the answer is just $1/12$.

(d) Compute the correlation coefficient $\rho(X, Z)$. **ANSWER:**

$$\frac{\mathrm{Cov}(X,Z)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Z)}} = \frac{\mathrm{Cov}(X, X+Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(X+Y)}} = \frac{\mathrm{Var}(X)}{\sqrt{\mathrm{Var}(X) 2\mathrm{Var}(X)}} = \frac{1}{\sqrt{2}}.$$

7. (10 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on the interval $[0,1]$. Write $X = X_1 + X_2 + \ldots + X_n$.

(a) Compute the characteristic function $\phi_{X_1}(t)$. **ANSWER:** $\phi_{X_1}(t) = E[e^{itX_1}] = \int_0^1 e^{itx} dx = e^{itx}/it\Big|_0^1 = (e^{it} - 1)/it$

(b) Compute the characteristic function $\phi_X(t)$. **ANSWER:** $\phi_X(t) = \left(\phi_{X_1}(t)\right)^n$.

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