NAME: ______________________________

Spring 2014 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Let X be a uniformly distributed random variable on $[-1, 1]$.

(a) Compute the variance of X^2.

(b) If $X_1, \ldots, X_n$ are independent copies of X, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function F_Z?

2. (10 points) A certain bench at a popular park can hold up to two people. People in this park walk in pairs or alone, but nobody ever sits down next to a stranger. They are just not friendly in that particular way. Individuals or pairs who sit on a bench stay for at least 1 minute, and tend to stay for 4 minutes on average. Transition probabilities are as follows:

(i) If the bench is empty, then by the next minute it has a $1/2$ chance of being empty, a $1/4$ chance of being occupied by 1 person, and a $1/4$ chance of being occupied by 2 people.

(ii) If it has 1 person, then by the next minute it has $1/4$ chance of being empty and a $3/4$ chance of remaining occupied by 1 person.

(iii) If it has 2 people then by the next minute it has $1/4$ chance of being empty and a $3/4$ chance of remaining occupied by 2 people.

(a) Use E, S, D to denote respectively the states empty, singly occupied, and doubly occupied. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by E, S, and D.

(b) If the bench is empty, what is the probability it will be empty two minutes later?

(c) Over the long term, what fraction of the time does the bench spend in each of the three states?

3. (10 points) Eight people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 8! permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N]$

(b) $P(N = 7)$

(c) $P(N = 0)$

4. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 5 with probability 1/2 and -5 with probability 1/2. Write $Y_n = \sum_{i=1}^n X_i$. Answer the following:

(a) What is the probability that Y_n reaches 65 before the first time that it reaches -15?

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = 5X_n$

(ii) $Z_n = 5^{-n} \prod_{i=1}^n X_i$

(iii) $Z_n = \prod_{i=1}^n X_i^2$

(iv) $Z_n = 17$

(v) $Z_n = X_n - 4$

5. (10 points) Suppose that X and Y are independent exponential random variables with parameter $\lambda = 2$. Write $Z = \min\{X, Y\}$

(a) Compute the probability density function for Z.

(b) Express $E[\cos(X^2Y^3)]$ as a double integral. (You don't have to explicitly evaluate the integral.)

6. (10 points) Let X_1, X_2, X_3 be independent standard die rolls (i.e., each of $\{1, 2, 3, 4, 5, 6\}$ is equally likely). Write $Z = X_1 + X_2 + X_3$.

(a) Compute the conditional probability $P[X_1 = 6 | Z = 16]$.

(b) Compute the conditional expectation $E[X_2 | Z]$ as a function of Z (for $Z \in \{3, 4, 5, \ldots, 18\}$).

7. (10 points) Suppose that X_i are i.i.d. uniform random variables on $[0, 1]$.

(a) Compute the moment generating function for X_1.

(b) Compute the moment generating function for the sum $\sum_{i=1}^{n} X_i$.

8. (10 points) Let X be a normal random variable with mean 0 and variance 5.

(a) Compute $\mathbb{E}[e^X]$.

(b) Compute $\mathbb{E}[X^9 + X^3 - 50X + 7]$.

9. (10 points) Let X and Y be independent random variables. Suppose X takes values $\{1, 2\}$ each with probability $1/2$ and Y takes values $\{1, 2, 3\}$ each with probability $1/3$. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$.

(b) Compute $H(X, Z)$.

(c) Compute $H(2^X 3^Y)$.

10. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability 1/3 and .5 with probability 2/3. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^n X_i$ for $n \geq 1$.

(a) What is the the probability that Y_n reaches 4 before the first time that it reaches $\frac{1}{64}$?

(b) Find the mean and variance of $\log Y_{400}$.

(c) Compute $\mathbb{E}Y_{100}$.

18.600 Probability and Random Variables
Fall 2019