

18.465 PS6 due Thursday, April 14, 2005

1. Let P_4 put probability $1/4$ at each of the vertices of a parallelogram in the plane. Let m be an affinely equivariant location functional (not necessarily singularly affine equivariant) defined at P_4 . Show that $m(P_4)$ is the sample mean of the vertices. *Hint:* prove it first for a square centered at the origin, then for a rectangle, then for a general parallelogram via non-singular affine transformations.
2. Consider the 6 points of \mathbb{R}^2 : $(0, -1)$, $(0, 1)$, $(1, 6)$, $(1, -8)$, $(3, -1)$, $(3, 1)$. Find the minimum-volume (in this case minimum-area) ellipsoid (ellipse) functionals for this data set (a location vector μ and a matrix C), as defined on pp. 7-8 of the April 7 handout. *Hints:* the ellipse is $(x - \mu)'C^{-1}(x - \mu) \leq 1$. For $d = 2$, let $k = \lfloor n/2 \rfloor + 1 = \lfloor (n + d + 1)/2 \rfloor$, which equals what for $n = 6$? Find the ellipse of smallest area containing k of the points. A given rectangle is inscribed in many ellipses. To find the one of minimum area you can easily guess what it is when the rectangle is a square, then transform it affinely. To check it you can do a calculation with a Lagrange multiplier.
3. For the 6 points in \mathbb{R} , $-6, -5, -1, 1, 5, 6$, the shortest interval containing 4 of the points is not unique. What are the two shortest such intervals?
4. In the minimum-volume-ellipsoid functional, find c_d (defined on p. 8 of the April 7 handout) for $d = 1$ and $d = 2$. *Hints:* for a standard normal distribution $N(0, I)$, c_d is the square of the radius of the ball B centered at 0 with $N(0, I)(B) = 1/2$. (The ball is an interval for $d = 1$ and a disk for $d = 2$.) Find the radius for $d = 1$ from normal quartiles and for $d = 2$ using polar coordinates.
5. Let P_3 put probability $1/3$ each in the points $(-1, 0)$, $(0, 3)$, and $(1, 0)$. Show that every affinely equivariant location functional m defined at P_3 has the same value $m(P_3)$ and evaluate it. Is it the same as the spatial median for P_3 ?