18.440 Midterm 1, Spring 2011: 50 minutes, 100 points

1. Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.

2. No calculators, books, or notes may be used.

3. Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).

1. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p.

(a) Let X be such that the first heads appears on the Xth toss. In other words, X is the number of tosses required to obtain a heads. Compute (in terms of p) the expectation $E[X]$.

(b) Compute (in terms of p) the probability that exactly 5 of the first 10 tosses are heads.

(c) Compute (in terms of p) the probability that the 5th head appears on the 10th toss.

2. (20 points) Jill sends her resume to 1000 companies she finds on monster.com. Each company responds with probability 3/1000 (independently of what all the other companies do). Let R be the number of companies that respond.

(a) Compute $E[R]$. (Give an exact answer, not an approximate one.)

(b) Compute $\text{Var}[R]$. (Give an exact answer, not an approximate one.)

(c) Use a Poisson random variable approximation to estimate the probability $P\{R = 3\}$.

3. (10 points) How many four-tuples (x_1, x_2, x_3, x_4) of *non-negative* integers satisfy $x_1 + x_2 + x_3 + x_4 = 10$?

4. (10 points) Suppose you buy a lottery ticket that gives you a one in a million chance to win a million dollars. Let X be the amount you win. Compute the following:

(a) $E[X]$.

(b) $\text{Var}[X]$.

5. (20 points) Suppose that X is continuous random variable with
probability density function $f_X(x) = \begin{cases} 2x & x \in [0, 1] \\ 0 & x \notin [0, 1] \end{cases}$. Compute the
following:

(a) The expectation $E[X]$.

(b) The variance $\text{Var}[X]$.

(c) The cumulative distribution function F_X.

6. (20 points) A standard deck of 52 cards contains 4 aces. Suppose we choose a random ordering (all 52! permutations being equally likely). Compute the following:

(a) The probability that *all* of the top 4 cards in the deck are aces.

(b) The probability that *none* of the top 4 cards in the deck is an ace.

(c) The *expected* number of aces among the top 4 cards in the deck. (There is a simple form for the solution.)

18.440 Probability and Random Variables
Spring 2014