18.433 Combinatorial Optimization Due: Nov 18

Assignment 3: Convex Programming

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1. A meat packing plant produces 480 hams, 400 pork bellies and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies and picnics that can be smoked during a normal working day is 420; in addition, up to 250 products can be smoked on overtime at a higher cost. The *net* profits are as follows:

	Fresh	*Smoked(regular)*	*Smoked(overtime)*
Hams	\$8	\$14	\$11
Bellies	\$4	\$12	\$7
Picnics	\$4	\$13	\$9

   Find the production schedule that maximizes the total net profit, and give a proof that it is the optimal one.

2. Give a pivot rule for the Simplex algorithm which can lead to cycling and an example showing this.

3. Gordon's lemma says that for any $n \times n$ matrix A, exactly one of the following holds:

   - $Ax = 0$ for some $x \neq 0$, $x \geq 0$.
   - $y^T A < 0$ for some y.

   Here x and y are vectors in $\mathbf{R}^n$. Prove the lemma and give a geometric interpretation.

4. Let K be a convex body (a closed, bounded, convex set) and x be a point. Show that there is a unique point $y \in K$ which minimizes the Euclidean distance from x to some point in K (assume that there exists such a point).

5. Volume computation.

   (a) (Bonus) Let S be a convex body in $\mathbf{R}^n$. Let A be an $n \times n$ nonsingular matrix (i.e. $det(A) \neq 0$). Show that the set $\{Ax | x \in S\}$ is also convex and that its volume is $|det(A)| \cdot vol(S)$.

(b) An ellipsoid in $\mathbf{R}^n$ is the set of points $\{x|(Ax)^T(Ax) \leq r^2\}$ for some nonsingular $n \times n$ matrix A. Show that the volume of an ellipsoid is $vol(B_{n,r})/|det(A)|$, where $B_{n,r}$ is the n-dimensional ball of radius r.

6. Given a complete graph $G = (V, E)$ with a positive length w_{ij} between each pair of vertices $i, j \in V$, the traveling salesman problem is to find a minimum length Hamilton cycle of G.

   (a) Give an integer linear program to solve the traveling salesman problem.

   (b) Relax the integrality constraints to obtain a linear programming relaxation. Show that the linear program can be solved in polynomial-time by designing a separation oracle.