

# Lecture 22: Lévy Distributions

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## 1 Central Limit Theorem (from previously)

Let  $\{x_i\}$  be IID random variables with finite mean  $\mu$  and finite variance  $\sigma^2$ . Then the RV

$$S_n = \frac{1}{\sigma\sqrt{n}} \left( \sum_{j=1}^n x_j - n\mu \right) \quad (1)$$

converges to a standard  $N(0, 1)$  Gaussian density.

## 2 Introduction

This lectures covers similar material to Lectures 12 and 13 in the 2003 lecture notes. Also, see Hughes<sup>1</sup> §4.3.

Motivating question: What are the limiting distributions for a sum of IID variables? (And, in particular, those who violate the assumptions of the CLT.)

## 3 Lévy stability laws

Definitions:  $\{x_i\}$  RVs and  $X_N = \sum_{n=1}^N x_n$

Characteristic function:

$$\hat{p}_n(k) = [\hat{p}(k)]^n \quad (2)$$

$$\hat{p}_{n \times m}(k) = [\hat{p}_m(k)]^n = [\hat{p}(k)]^{n \times m} \quad (3)$$

Introduce new variables  $Z_N = \frac{X_N}{a_N}$ . The PDF of the  $Z_N$ 's is given by  $F_N(x) = p_N \frac{x}{a_N} \frac{1}{a_N}$ .

$$\hat{F}_N(a_N k) = \hat{p}_N(k) \quad (4)$$

Substituting in,

$$\hat{F}_{n \times m}(a_{nm}k) = \left[ \hat{F}_m(a_m k) \right]^n \quad (5)$$

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<sup>1</sup>B. Hughes, *Random Walks and Random Environments*, Vol. I, Sec. 2.1 (Oxford, 1995).

As  $n \rightarrow \infty$ , we want  $\hat{F}_n(k)$  to tend to some fixed distribution  $\hat{F}(k)$ . Say that  $\lim_{m \rightarrow \infty} \frac{a_{nm}}{a_m} = c_n$ ,

$$\hat{F}(kc_n) = \left[ \hat{F}(k) \right]^n \tag{6}$$

Suppose I want to find

$$\hat{F}(k\mu(\lambda)) = \left[ \hat{F}(k) \right]^\lambda \tag{7}$$

Introduce  $\psi(k) = \log \hat{F}(k)$

$$\psi(k\mu(\lambda)) = \lambda\psi(k) \tag{8}$$

Differentiate:

$$k\mu'(\lambda)\psi'(k\mu(\lambda)) = \psi(\lambda) \tag{9}$$

$$\mu(\lambda = 1) = 1 \tag{10}$$

$$k\mu'(1)\psi'(k) = \psi(k) \tag{11}$$

$$\frac{d\psi}{dk} = \frac{\psi}{\mu'(1)k} \tag{12}$$

This has a solution:

$$\psi(k) = \begin{cases} v_1 |k|^\alpha & k > 0 \\ v_2 |k|^\alpha & k < 0 \end{cases} \tag{13}$$

$$\hat{F}(k) = \begin{cases} \exp(v_1 |k|^\alpha) & k > 0 \\ \exp(v_2 |k|^\alpha) & k < 0 \end{cases} \tag{14}$$

$$v_1, v_2 \in \mathcal{C} \tag{15}$$

Thus we have this condition:

$$\hat{F}(-k) = \hat{F}(k)^* \tag{16}$$

$$\implies \hat{F}(k) = \exp((c_1 + ic_2 \operatorname{sgn}(k)) |k|^\alpha) \tag{17}$$

$$c_1, c_2 \in \Re \tag{18}$$

We can also write:

$$\hat{F}(k) = \exp \left[ -a |k|^\alpha \left( 1 - i\beta \tan \frac{\alpha\pi}{2} \operatorname{sgn}(k) \right) \right] \tag{19}$$

In this equation,  $\beta$  relates to the skewness of the distribution.

## 4 Lévy distribution

For above  $\hat{F}$  when  $\beta = 0$  (ie. symmetric limiting distribution), you get the Lévy distribution. It is defined in terms of its Fourier transform.

$$\hat{L}_\alpha(a, k) = \hat{F}(k) = \exp(-a |k|^\alpha) \tag{20}$$

Two special cases of the the Lévy distribution are  $\alpha = 1, 2$ .

$$\alpha = 2 \quad \hat{F}(k) = \exp(-ak^2) \quad \text{like Gaussian but more general} \tag{21}$$

$$\alpha = 1 \quad \hat{F}(k) = \exp(-a |k|) \quad \text{Cauchy} \tag{22}$$

### 4.1 Large x expansions of the Lévy distribution

$$L_\alpha(a, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp ikx - a |k|^\alpha dk \tag{23}$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos kx \exp -ak^\alpha dk \tag{24}$$

Integrate by parts: (25)

$$= \frac{a\alpha}{\pi} \int_0^{\infty} \sin \frac{k|x|}{|x|} \exp -ak^\alpha k^{\alpha-1} dk = \frac{a\alpha}{\pi |x|^{1+\alpha}} \int_0^{\infty} \xi^{\alpha-1} \sin \xi \exp \frac{-a\xi^\alpha}{|x|^\alpha} d\xi \tag{26}$$

Where we have made the following substitutions:

$$\xi = k |x|, d\xi = dk |x| \tag{27}$$

Taking the limit  $x \rightarrow \infty$ :

$$\sim_{x \rightarrow \infty} \frac{a\alpha}{\pi |x|^{1+\alpha}} \int_0^{\infty} \xi^{\alpha-1} \sin \xi d\xi \tag{28}$$

$$\sim \frac{a\alpha \Gamma(\alpha) \sin \frac{\pi\alpha}{2}}{\pi |x|^{1+\alpha}} \quad \text{power-law tail} \tag{29}$$

### 4.2 Small x expansion

$$L_\alpha(a, x) = \frac{1}{\pi} \int_0^{\infty} \cos kx \exp -ak^\alpha dk \tag{30}$$

Taylor expand cosine: 
$$= \frac{1}{\pi} \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n}}{(2n)!} \exp -ak^\alpha dk \tag{31}$$

Substitute:  $w = ak^\alpha \quad dw = \alpha ak^{\alpha-1} dk$  (32)

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} (-1)^n \frac{(x)^{2n}}{(2n)!} \frac{w^{2n/\alpha}}{a} \exp -w \frac{dw}{\alpha a \frac{w}{\alpha}} \tag{33}$$

We're only interested in evaluating this equation as  $x \rightarrow 0$ :

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(x)^{2n}}{a \frac{2n+1}{\alpha}} \Gamma\left(\frac{2n+1}{\alpha}\right) \tag{34}$$

$$\sim \frac{1}{\pi} \frac{\Gamma(\frac{1}{\alpha})}{\alpha a^{1/\alpha}} \quad \text{increases as } \alpha \rightarrow 0 \tag{35}$$

## 5 Analogy with Central Limit Theorem?

### 5.1 Gnedenko-Doblin Theorem

$\frac{1}{a_N} p_N \frac{x}{a_N} \rightarrow L_{\alpha,\beta}(a, x)$  iff the CDF  $P(x)$  satisfies

- $\lim_{x \rightarrow \infty} \frac{p(-x)}{a-p(x)} = \frac{1-\beta}{1+\beta}$

This is akin to the “amount” of probability on one side of the tail.

$$2. \lim_{x \rightarrow \infty} \frac{1-p(x)+p(-x)}{1-p(rx)+p(-rx)} = r^\alpha \quad \forall r$$

We can think of this as saying how fast it is decaying.

### 5.2 Theorem

The distribution  $p(x)$  is in the basin of attraction of  $L_{\alpha,\beta}(a, x)$  if

$$p(x) \sim \frac{A_\pm}{|x|^{1+\alpha}} \quad x \rightarrow \pm\infty \tag{36}$$

$$\beta = \frac{A_+ - A_-}{A_+ + A_-} \tag{37}$$

If  $p(x)$  has power-law tails, then the sum of RVs goes to a Lévy distribution.

### 5.3 Scaling and superdiffusivity

Suppose that  $\{x_i\}$  follow a  $L_\alpha(a, x)$  distribution, then  $\hat{p}(k) = \exp -a |k|^\alpha$  and then the characteristic function at  $x_N = \sum_{n=1}^N x_n$  is given by

$$\hat{p}_N(k) = [\exp -a |k|^\alpha]^N = \exp -a |k|^\alpha N \tag{38}$$

$$p_N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp ikx - aN |k|^\alpha dk \tag{39}$$

Now we substitute:

$$k = VN^{-1/\alpha}, dk = -dVN^{-1/\alpha} \tag{40}$$

$$p_N(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp iV \left( \frac{x}{N^{1/\alpha}} \right) - a|V|^\alpha \frac{dV}{N^{1/\alpha}} = \frac{1}{N^{1/\alpha}} L_\alpha \left( a, \frac{x}{N^{1/\alpha}} \right) \tag{41}$$

Thus, the width scales like  $N^{1/\alpha}$ .

If  $\alpha > 2$ , we get scaling that exceeds “square-root” scaling, ie. a superdiffusive process.