18.366 Random Walks and Diffusion, Spring 2005, M. Z. Bazant.

# Problem Set 2

Due at lecture on Tuesday, March 1, 2005.

1. **Percentile order statistics.** Let $X_i$ ($i = 1, ..., N$) be IID continuous random variables with CDF, $P(x) = \text{Prob}(X_i \le x)$, and PDF, $p(x) = dP/dx$. Let $Y_N^{(\alpha)}$ be the $100\alpha$th percentile, which is uniquely defined by ordering the outcomes, $X_{(1)} < X_{(2)} < \ldots < X_{(N)}$ (using the standard notation of *order statistics*[1]) and setting $Y_N^{(\alpha)} = X_{([\alpha N])}$ (where $[x]$ is the nearest integer to $x$).

   (a) Derive an expression for the PDF $f_N(y)$ of $Y_N^{(\alpha)}$ which is related to the PDF of a Bernoulli random walk with probability $P(y)$ of 'stepping to the right'. Explain why the PDF is sampled in the tail, beyond the 'central region'.

   (b) Using the saddle-point analysis of the Bernoulli random walk, derive a globally-valid asymptotic approximation for $f_N(y)$ as $N \to \infty$. (First replace $[\alpha N]$ by $\alpha N$, which can be rigorously justified in this limit.)

   (c) Show that a sort of 'CLT' holds: The variable $(Y_n - y_\alpha)/\sigma_{Y_N}$ has an asymptotic normal distribution, where the mean is $y_\alpha$, the parent percentile defined by $P(y_\alpha) = \alpha$, and the variance is
$$\sigma_{Y_N}^2 = \frac{\alpha(1-\alpha)}{Np(y_\alpha)^2}.$$

   (d) Suppose that $X_i$ are iid standard normal random variables (mean 0, variance 1). Compare the limiting distributions of the mean $Z_N = \sum_{n=1}^N X_n/N$ and the median $Y_N^{(0.5)}$. Which has a bigger variance?

2. **Winding angle of Pearson's walk.** Simulate Pearson's random walk in the plane with IID steps of constant length and uniform random angle. Define the "winding angle", $\Theta_N$, to be the polar angle of the walker's position after $N$ steps, measured from the angle of the first step (so that $\langle \Theta_N \rangle = 0$ for all $N \ge 1$). The winding angle is allowed to vary continuously in the range, $-\infty < \Theta_N < \infty$ in order to count how many times the walker encircles the origin[2].

   (a) Compile statistics to check Berger's formula:
$$\langle \Theta_N^2 \rangle = \frac{1}{4}\log^2 N + \frac{4-\gamma}{2}\log N + O(1)$$
   where $\gamma = 0.5772\ldots$ is Euler's constant. Plot $\sigma_N = \sqrt{\langle \Theta_N^2 \rangle}$ versus $\log N$, and compare with this result.

   (b) Plot a histogram of the PDF of $\Theta_N$ for a large value of $N$, and try to fit to an analytical form.

[1] Order statistics have many applications. For example, the MIT admissions committee may wonder whether or not GRE Mathematics test scores from different years can be meaningfully compared. This could be done by seeing whether various percentiles of actual scores are consistent with the hypothesis of stationary, random fluctuations from year to year. Order statistics are also used to price Internet bandwidth, billed in proportion to the 95th percentile of data-rate samples taken at regular intervals over a one month billing period (e.g. $500/(Mbit/sec).)

[2] The winding problem, first considered by Paul Lévy in 1940, could describe the wrapping of a polymer molecule around a line, or the trapping of a flux tube on a columnar defect in a Type II superconductor.

3. **Globally valid asymptotics.** Consider the PDF,

$$p(x) = \frac{1}{2a} e^{-|x|/a}$$

for IID displacements in a one-dimensional random walk. Use the saddle-point method to derive a globally-valid asymptotic approximation as $N \to \infty$ for the PDF of the position after $N$ steps in terms of the scaled variable $\xi = x/N$.

4. **The Void Model.** In a simple two-dimensional model to describe granular drainage in a silo [3], particles are arranged on rectangular lattice, with positions labeled by $(M, N)$, where $M$ and $N$ are integers and $N \geq 0$. To create flow in the silo, voids of empty space are introduced at the silo base, at $N = 0$, which then move upwards according to a simple random walk. If a void is at $(M, N)$, it has an equal chance of moving to up-left to $(M-1, N)$, or up-right to $(M+1, N)$. Each time a void moves to a new location, the particle there is displaced downwards to the void's previous position.

   (a) Let $V_N(M)$ be the probability of finding a single void at horizontal position $M$, given that it is at height $N$. Find a recurrence relation for the probability distribution $V_{N+1}(M)$ in terms that for $V_N(M)$. Let the horizontal and vertical lattice spacings be $d$ and $h$ respectively. In the continuum limit ($N \gg 1$), show that the void density, $\eta(x, y) = \eta(Md, Nh) = V_N(M)$, which varies on scales much larger than $h, d$, satisfies a diffusion equation,

   $$\eta_y = b\eta_{xx}$$

   What is the "diffusion length", $b$?

   (b) Assume that voids pass independently through the material without interacting, and consider the dynamics of a tracer particle in the flow. Let $P_N(M)$ be the conditional probability that the particle is at horizontal position $M$ when it falls to a height $N$, and show that $P_N(M)$ satsifies the following recursion:

   $$P_{N-1}(M) = P_N(M-1)\frac{V_{N-1}(M)}{2V_N(M-1)} + P_N(M+1)\frac{V_{N-1}(M)}{2V_N(M+1)}.$$

   (c) In the continuum limit, show that $\rho(x, y) = \rho(Md, Nh) = P_N(M)$, satisfies the partial differential equation,

   $$\rho_y = 2b\frac{\partial}{\partial x}\left(\frac{\rho\eta_x}{\eta}\right) - b\rho_{xx}.$$

   (d) Solve the equation for $\eta$ in the domain $y > 0$, given the initial condition $\eta(x, 0) = \delta(x)$. For this case, solve the resulting equation for $\rho$, e.g. by taking the Fourier transform with respect to $x$ and using the method of characteristics.

[3] The Void Model also describes diffusion in crystals, where the microscopic mechanism is more realistic.