## 17    Towards airplane flight

There are two forces, lift and drag, experienced when an airplane wing moves through the air. At the start of the 20th century, however, fluid dynamicists were unable to correctly predict them. In fact, the lift and drag were determined to be identically zero, for all wing shapes (despite the fact that the Wright brothers had successfully built airplanes!). To proceed further into airplane flight, and how this problem was resolved, we shall first need some fluid mechanical preliminaries.

### 17.1    High-Reynolds number limit

We have already derived the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \boldsymbol{u} \,, \qquad \nabla \cdot \boldsymbol{u} = 0. \tag{433}$$

The ratio of the viscous forces to inertial forces is

$$\frac{\nu \nabla^2 \boldsymbol{u}}{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} \sim \frac{\nu U / L^2}{U^2 / L} = \frac{\nu}{UL} = \frac{1}{\mathrm{Re}} \tag{434}$$

where Re is the *Reynolds number*. For an airplane typical values are $U$=400mph, $L$=5m and $\nu$=0.1cm$^2$/s, giving $Re = 10^8$. Thus, the inertial forces are eight orders of magnitude larger than viscous forces, so it is seems very reasonable that we can neglect them. Doing so, we are left with the *Euler equations* for an inviscid fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} \,, \qquad \nabla \cdot \boldsymbol{u} = 0. \tag{435}$$

We shall first convince ourselves that these equations predict both the lift force ($F_l$) and the drag force ($F_d$) to be zero, independent of the shape of the wing.

The Euler equations can be written in an interesting form using the vector identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \frac{\nabla(u^2)}{2} \tag{436}$$

as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right). \tag{437}$$

where $u$ is the magnitude of $\boldsymbol{u}$. The quantity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{438}$$

is called the *vorticity*. In 2D it represents the average angular velocity of the fluid around any point, and has little to do with global rotations of the fluid (e.g., shear flows have non-zero vorticity). If the flow is steady ($\partial \boldsymbol{u}/\partial t = \boldsymbol{0}$) and *irrotational* ($\boldsymbol{\omega} = \boldsymbol{0}$) then we have *Bernoulli's law*:

$$H = \frac{p}{\rho} + \frac{u^2}{2} = constant. \tag{439}$$

The constancy of $H$ in irrotational flow is a famous result, and has many simple qualitative consequences. It requires that the pressure in a fluid is smaller when the velocity is larger, and is a statement of conservation of energy when viscous dissipation can safely be ignored.

What are the consequences of $\boldsymbol{\omega} = \boldsymbol{0}$? The most important is that we can look for solutions $\boldsymbol{u}$ of the Euler equation that can be derived from velocity potential:

$$\boldsymbol{u} = \nabla\phi. \tag{440}$$

Now the wonderful thing is that if the flow is also incompressible, then from $\nabla \cdot \boldsymbol{u} = 0$ we obtain

$$\nabla^2\phi = 0, \tag{441}$$

and we have reduced the problem to solving Laplace's equation. We can determine the pressure from the *Bernoulli relation*

$$\frac{p}{\rho} = H - \frac{(\nabla\phi)^2}{2}. \tag{442}$$

Solutions of these equations are typically called *ideal flows*, because viscosity has been neglected.

## 17.2   Kelvin's Theorem

When a plane is at rest we can reasonably argue that flow is initially irrotational ($\boldsymbol{\omega} = \boldsymbol{0}$). When the system becomes non-steady, and the plane accelerates, what happens? To answer this, consider the *circulation* around a closed loop moving with the flow:

$$\Gamma = \int_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{l}. \tag{443a}$$

By Stoke's theorem

$$\Gamma(t) = \int \boldsymbol{\omega} \cdot \boldsymbol{n} dA. \tag{443b}$$

Thus if $\boldsymbol{\omega} = \boldsymbol{0}$, $\Gamma = 0$. What is the time evolution of $\Gamma$? We know that

$$\frac{D\Gamma}{Dt} = \frac{d}{dt} \int_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{l} = \int \left[ \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{l} + \boldsymbol{u} \cdot \frac{D(d\boldsymbol{l})}{Dt} \right]. \tag{444}$$

The first term concerns changes in the velocity field and is identically zero since

$$\int \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{l} = \frac{1}{\rho} \int -\nabla p \cdot d\boldsymbol{l} = \frac{1}{\rho} \int (\nabla \times \nabla p) \cdot \boldsymbol{n} dA = 0. \tag{445}$$

The second term relates to stretching of the loop and is zero for essentially geometric reasons. One has that

$$\frac{D(d\boldsymbol{l})}{Dt} = d\boldsymbol{u} \tag{446}$$

since one end of the spatial vector $d\boldsymbol{l}$ moves with velocity $\boldsymbol{u}$ and the other with velocity $\boldsymbol{u} + d\boldsymbol{u}$. Thus

$$\int \boldsymbol{u} \cdot \frac{D(d\boldsymbol{l})}{Dt} = \int \boldsymbol{u} \cdot d\boldsymbol{u} = \int \frac{d(u^2)}{2} = 0, \tag{447}$$

because the integration is around a closed loop and $u^2$ has the same value at the start and end of it. But this implies

$$\frac{D\Gamma}{Dt} = 0, \tag{448}$$

so the circulation around a loop remains constant. If it is initially zero then it remains zero for all subsequent times. We should keep in mind, however, that our derivations of this result relied on the assumption that the fluid is inviscid and incompressible[24]

One of the most interesting things about this theorem is its historical origin: Kelvin viewed it as the basis of his vortex theory of the atom.[25]

---

[24]More generally, $D\Gamma/Dt = 0$ still holds for barotropic ideal fluids with conservative body forces.
[25]Read Acheson (p. 168) for an interesting discussion of this.

18.354J / 1.062J / 12.207J Nonlinear Dynamics II: Continuum Systems
Spring 2015