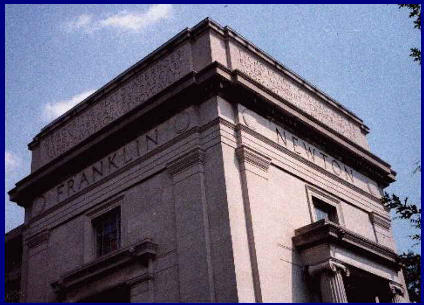
Advances in Random Matrix Theory: Let there be tools



Alan Edelman

Brian Sutton, Plamen Koev,

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Raj Rao and others

MIT: Dept of Mathematics,

Computer Science AI Laboratories

World Congress, Bernoulli Society

Barcelona, Spain

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Message

- Ingredient: Take Any important mathematics
- Then Randomize!
- This will have many applications!
- We can’t keep this in the hands of specialists anymore: Need tools!

Tools

- So many applications …
- Random matrix theory: catalyst for 21st century special functions, analytical techniques, statistical techniques
- In addition to mathematics and papers
  - Need tools for the novice!
  - Need tools for the engineers!
  - Need tools for the specialists!

Themes of this talk

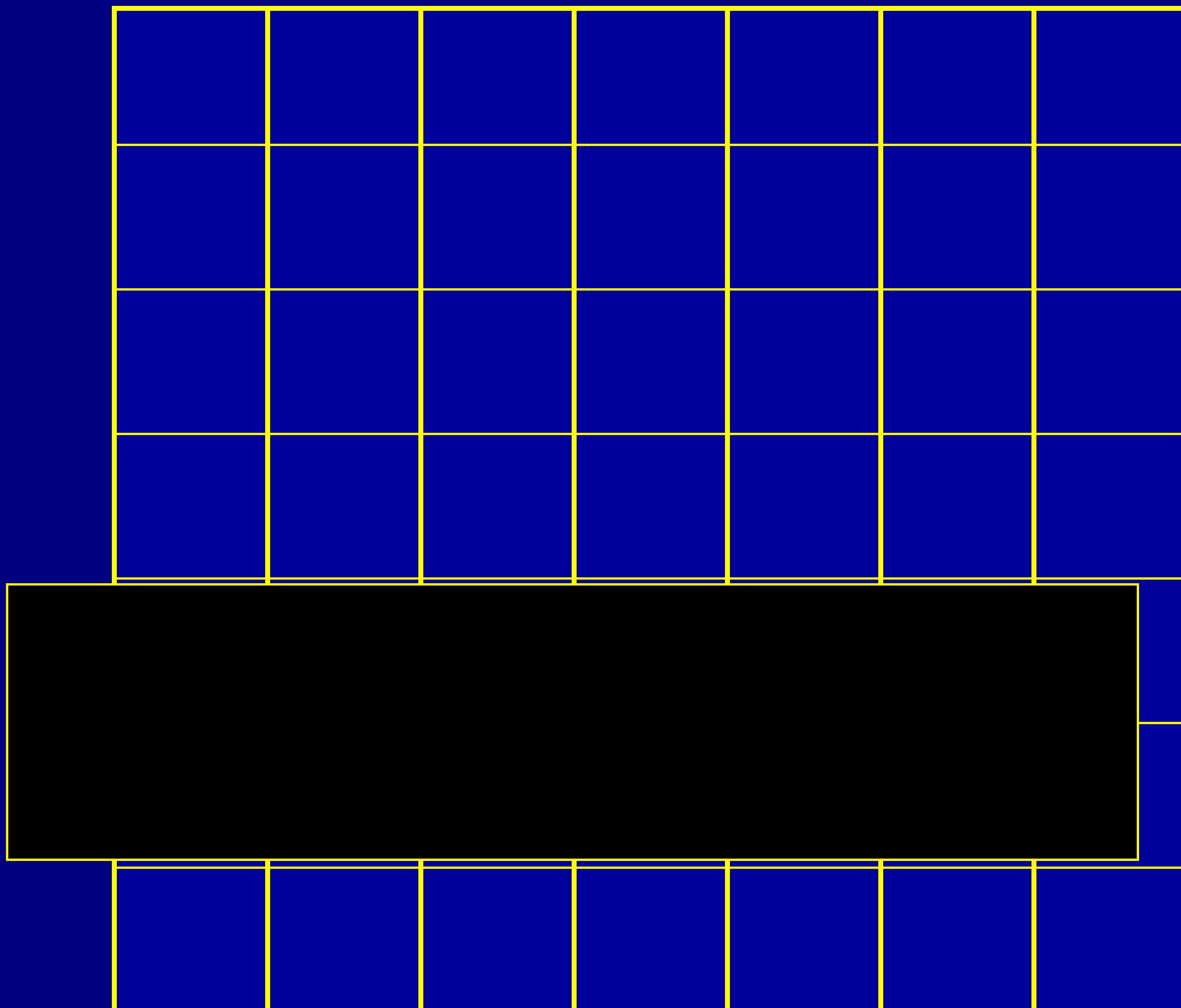
- Tools for general beta
  - What is beta? Think of it as a measure of (inverse) volatility in “classical” random matrices.
- Tools for complicated derived random matrices
- Tools for numerical computation and simulation

Wigner's Semi-Circle

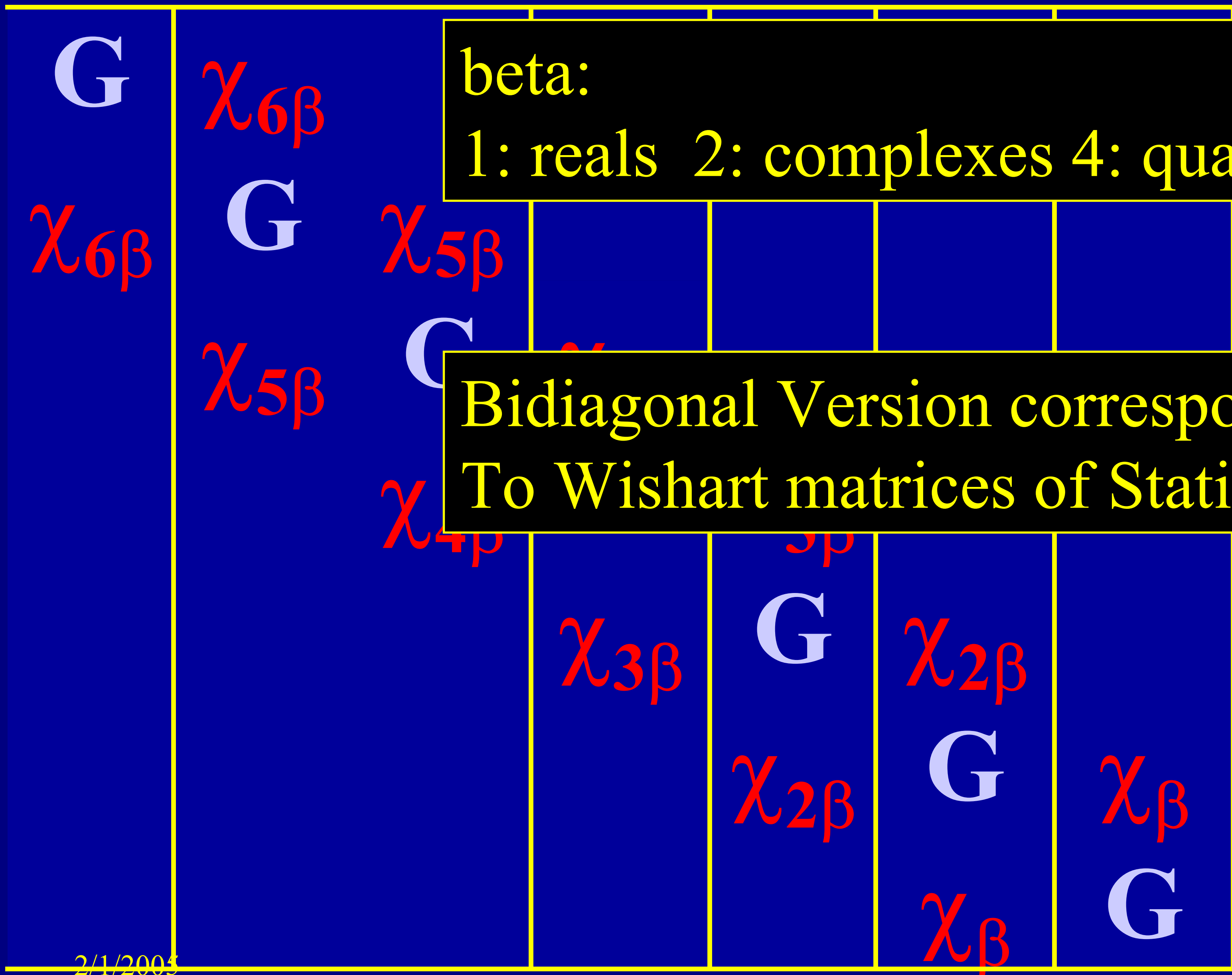
- The classical & most famous rand eig theorem
- Let S = random symmetric Gaussian
  - MATLAB: A=randn(n); S=(A+A')/2;
- S known as the Gaussian Orthogonal Ensemble
- Normalized eigenvalue histogram is a semi-circle

```
n=20;  s=30000; d=.05;     %matrix size, samples, sample dist
e=[];       %gather up eigenvalues
im=1;     %imaginary(1) or real(0)
for i=1:s,
  a=randn(n)+im*sqrt(-1)*randn(n);a=(a+a')/(2*sqrt(2*n*(im+1)));
  v=eig(a)'; e=[e v];
end
hold off; [m x]=hist(e,-1.5:d:1.5); bar(x,m*pi/(2*d*n*s));
axis('square'); axis([-1.5 1.5 -1 2]); hold on;
t=-1:.01:1; plot(t,sqrt(1-t.^2),'r');
```

sym matrix to tridiagonal form



General beta



beta:
1: reals 2: complexes 4: quaternions

Bidiagonal Version corresponds
To Wishart matrices of Statistics

Tools

- Motivation: A condition number problem
- Jack & Hypergeometric of Matrix Argument
- MOPS: Ioana Dumitriu's talk
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Numerical Analysis: Condition Numbers

- $\kappa(A)$ = "condition number of A"
- If $A=U\Sigma V'$ is the svd, then $\kappa(A) = \sigma_{max}/\sigma_{min}$.
- Alternatively, $\kappa(A) = \sqrt{\lambda_{max}(A'A)}/\sqrt{\lambda_{min}(A'A)}$
- One number that measures digits lost in finite precision and general matrix "badness"
  - Small=good
  - Large=bad
- The condition of a random matrix???

Von Neumann & co.

- Solve Ax=b via x= $(A'A)^{-1}A'$ b

  $M \approx A^{-1}$

- Matrix Residual: $||AM-I||_2$
- $||AM-I||_2 < 200\kappa^2 n \varepsilon$

  ↑ ≈

- How should we estimate κ?
- Assume, as a model, that the elements of A are independent standard normals!

Von Neumann & co. estimates (1947-1951)

- "For a 'random matrix' of order n the expectation value has been shown to be about ~~n~~" ∞

  Goldstine, von Neumann

- "… we choose two different values of κ, namely n and √10n" $P(\kappa<n) \approx 0.02$

  $P(\kappa<\sqrt{10n}) \approx 0.44$

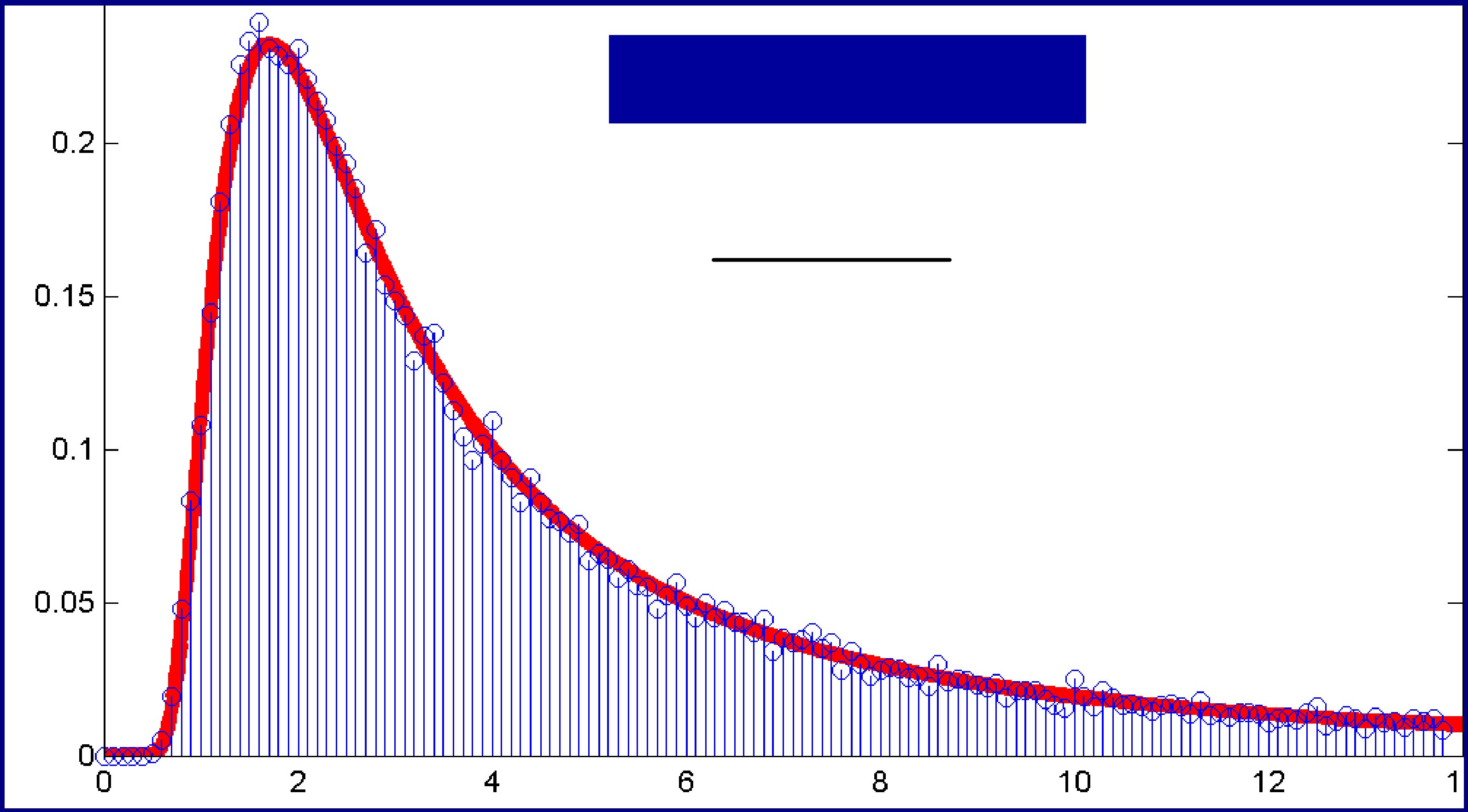
  Bargmann, Montgomery, vN

- "With a probability ~1 … κ < 10n"

  Goldstine, von Neumann

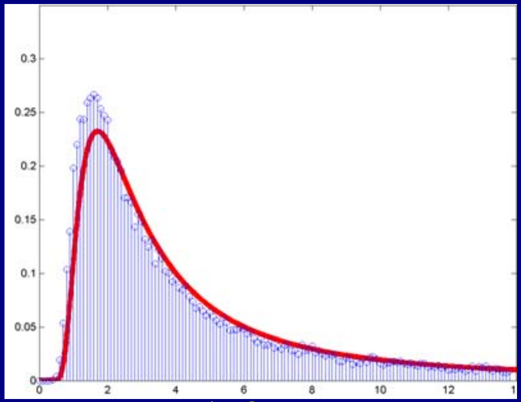
  $P(\kappa<10n) \approx 0.80$

Random cond numbers, n→∞

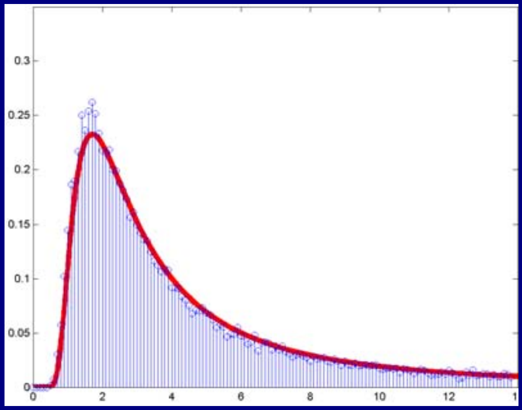


Experiment with n=200

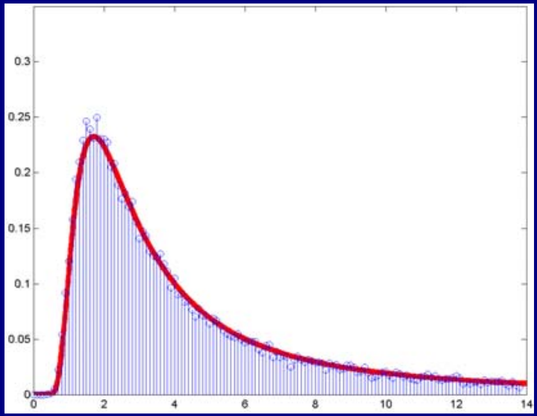
Finite n



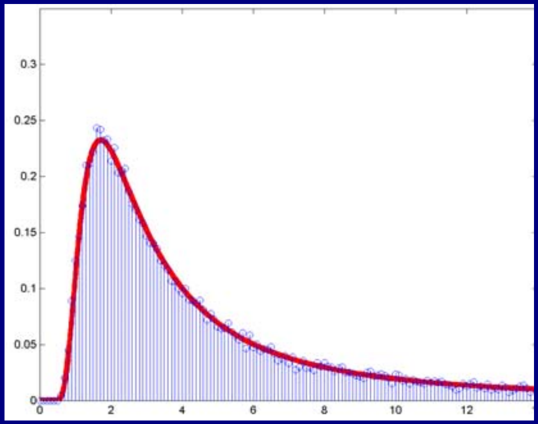
n=10



n=25

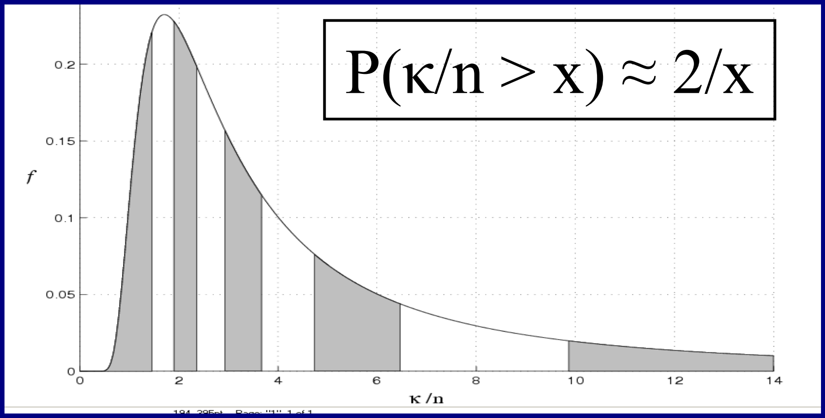


n=50

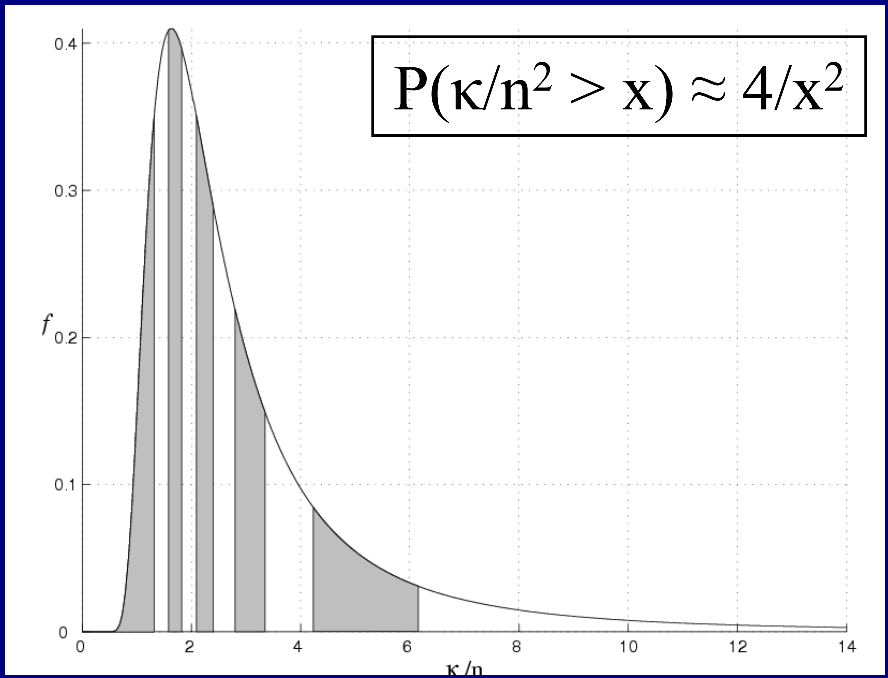


n=100

Condition Number Distributions



Real n x n, n→∞

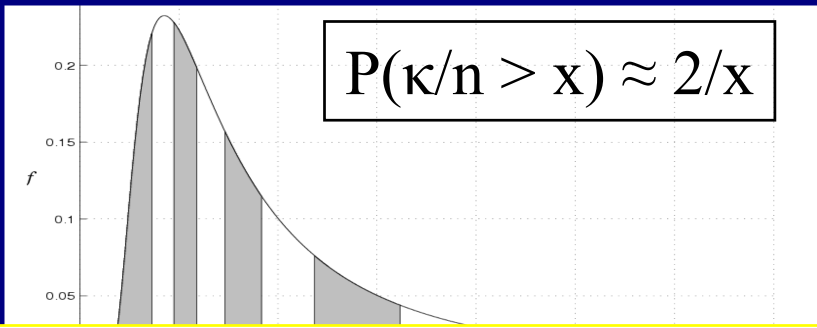


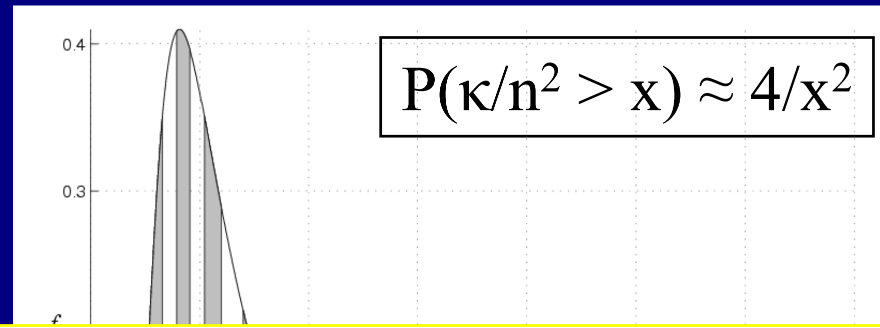
Complex n x n, n→∞

Generalizations:

- β: 1=real, 2=complex
- finite matrices
- rectangular: m x n

Condition Number Distributions





Square, $n \to \infty$: $P(\kappa/n^\beta > x) \approx (2\beta^{\beta-1}/\Gamma(\beta))/x^\beta$ (All Betas!!)

General Formula: $P(\kappa > x) \sim C\mu/x^{\beta(n-m+1)}$,

where $\mu = \beta(n-m+1)/2$th moment of the largest eigenvalue of $W_{m-1,n+1}(\beta)$

and C is a known geometrical constant.

Density for the largest eig of W is known in terms of ${}_1F_1((\beta/2)(n+1), ((\beta/2)(n+m-1); -(x/2)I_{m-1})$ from which μ is available

Tracy-Widom law applies probably all beta for large m,n.

Johnstone shows at least beta=1,2.

Tools

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Multivariate Orthogonal Polynomials & Hypergeometrics of Matrix Argument

- Ioana Dumitriu's talk
- The important special functions of the 21st century
- Begin with w(x) on I
  - $\int p_\kappa(x)p_\lambda(x)\ \Delta(x)^\beta \prod_i w(x_i)dx_i = \delta_{\kappa\lambda}$
  - Jack Polynomials orthogonal for w=1 on the unit circle. Analogs of x^m

Multivariate Hypergeometric Functions

- Univariate

$$_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) \equiv \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{k!(b_1)_k \ldots (b_q)_k} \cdot x^k,$$

where $(a)_k = a(a+1)\ldots(a+k-1)$.

- Hard problem to approximate. Slow convergence

Multivariate Hypergeometric Functions

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where $(a)_k = a(a+1)\ldots(a+k-1)$.

- Hard problem to approximate. Slow convergence
- Multivariate

$$_pF_q^{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1, \ldots, x_n) \equiv \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a_1)_\kappa \ldots (a_p)_\kappa}{k!(b_1)_\kappa \ldots (b_q)_\kappa} \cdot C_\kappa^\alpha(x_1, \ldots, x_n),$$

where

- $(a)_\kappa \equiv \prod_{(i,j)\in\kappa} \left(a - \frac{i-1}{\alpha} + j - 1\right)$ —Pocchammer symbol
- $C_\kappa^\alpha(x_1, x_2, \ldots, x_n)$ is the "C" Jack Function

An Application—p.d.f. of $\lambda_{\max}$ of a β-Laguerre matrix

Definition: $L = BB^T$, where

$$B = \begin{bmatrix} \chi_{2a} & & & \\ \chi_{\beta(n-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_{\beta} & \chi_{2a-\beta(n-1)} \end{bmatrix},$$

and $k = a - \frac{\beta}{2}(n-1) - 1$ is a nonnegative integer.

The p.d.f. of $\lambda_{\max}$ is

$$f(x) = x^{kn} \cdot e^{-\frac{nx}{2}} \cdot \ {}_2F_0^{2/\beta}(-k, \beta\frac{n}{2} + 1; ; -\frac{2}{x} I_{n-1}).$$

Wishart (Laguerre) Matrices!

Plamen's clever idea

Idea: Update, do not Compute

$$ {}_pF_q^{\alpha}(a_{1:p}; b_{1:q}; x_{1:n}) \approx \sum_{k=0}^{m} \sum_{\kappa \vdash k} \underbrace{\frac{(a_1)_\kappa \ldots (a_p)_\kappa}{k!(b_1)_\kappa \ldots (b_q)_\kappa}}_{Q_\kappa} \cdot C_\kappa^{\alpha}(x_{1:n}) $$

Store every computed Q_κ and C_κ^α.

If

$$ \begin{aligned} \kappa &= (\kappa_1, \kappa_2, \ldots, \kappa_i, \ldots) \\ \nu &= (\kappa_1, \kappa_2, \ldots, \kappa_i - 1, \ldots) \end{aligned} $$

Then

$$ \frac{Q_\kappa}{Q_\nu} = \frac{\prod_{j=1}^{p}(a_j + c)}{\prod_{j=1}^{q}(b_j + c)}, \quad \text{where} \quad c = -\frac{i-1}{\alpha} + \kappa_i - 1; $$

Cost: $2(p+q)$ instead of $|\kappa|(p+q)$.

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Mops (Dumitriu etc.) Symbolic

Maple 8 - [Untitled (1) - [Server 1]]

File Edit View Insert Format Spreadsheet Window Help

$libname$:= "F:\maple-work\SF", "F:\maple-work\Jack", "Z:\MAPLE8/lib"

```
Warning, the protected name conjugate has been redefined and unprotected
```

[*arm, conjugate, expjack, expjackj, expjackp, expjacks, gbinomial, ghypergeom, gsfact, hermite, issubpar, jack, jackcoeff, jackidentity, jackj, jackp, jacobi, jacobicoeff, laguerre, leg, lhook, par*, ρ, *sfact, subpar, tojack, uhook*]

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Warning, the name conjugate has been rebound
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[*Par, add_basis, char2sf, conjugate, dominate, dual_basis, evalsf, hooks, itensor, jt_matrix*, ω, *plethysm, scalar, sf2char, skew, subPar*, θ, *toe, toh, top, tos, varset, zee*]

```
> tojack(1, m[9],9);
```

$$C_{9}-\frac{1}{8}C_{8,1}+\frac{1}{28}C_{7,1,1}-\frac{1}{56}C_{6,1,1,1}+\frac{1}{70}C_{5,1,1,1,1}-\frac{1}{56}C_{4,1,1,1,1,1}+\frac{1}{28}C_{3,1,1,1,1,1,1}-\frac{1}{8}C_{2,1,1,1,1,1,1,1}+C_{1,1,1,1,1,1,1,1,1}$$

```
> tojack(1/2, m[9],9);
```

$$C_{9}-\frac{1}{4}C_{8,1}+\frac{1}{28}C_{7,2}+\frac{1}{7}C_{7,1,1}-\frac{1}{42}C_{6,2,1}-\frac{1}{7}C_{6,1,1,1}+\frac{1}{70}C_{5,2,2}+\frac{1}{35}C_{5,2,1,1}+\frac{8}{35}C_{5,1,1,1,1}-\frac{3}{140}C_{4,2,2,1}-\frac{2}{35}C_{4,2,1,1,1}-\frac{4}{7}C_{4,1,1,1,1,1}+\frac{1}{28}C_{3,2,2,2}+\frac{2}{35}C_{3,2,2,1,1}+\frac{4}{21}C_{3,2,1,1,1,1}+\frac{16}{7}C_{3,1,1,1,1,1,1}-\frac{1}{7}C_{2,2,2,2,1}-\frac{2}{7}C_{2,2,2,1,1,1}-\frac{8}{7}C_{2,2,1,1,1,1,1}-16\,C_{2,1,1,1,1,1,1,1}+256\,C_{1,1,1,1,1,1,1,1,1}$$

```
> tojack(1/3, m[9],9);
```

$$-\frac{27}{7}C_{2,2,2,2,1}+\frac{27}{140}C_{5,2,1,1}-\frac{1}{56}C_{6,3}+\frac{9}{28}C_{7,1,1}+\frac{81}{28}C_{3,2,1,1,1,1}-\frac{405}{56}C_{2,2,2,1,1,1}-\frac{3}{28}C_{6,2,1}-\frac{27}{56}C_{6,1,1,1}-\frac{729}{28}C_{2,2,1,1,1,1,1}-\frac{27}{112}C_{4,2,2,1}+\frac{1}{28}C_{3,3,3}+\frac{81}{70}C_{5,1,1,1,1}-\frac{81}{140}C_{4,2,1,1,1}+\frac{9}{112}C_{3,3,2,1}-\frac{243}{56}C_{4,1,1,1,1,1}-\frac{3}{112}C_{4,3,2}+\frac{3}{28}C_{7,2}+\frac{729}{28}C_{3,1,1,1,1,1,1}+C_{9}+\frac{3}{28}C_{5,2,2}-\frac{2187}{8}C_{2,1,1,1,1,1,1,1}$$

Symbolic MOPS applications

Applications: eigenvalue statistics

Examples for Hermite:

⇒ Expectation of traces of powers: [redacted]

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} \lambda_i^4\right) \prod_{i<j} |\lambda_i - \lambda_j|^{2/\alpha} \; e^{-\sum_{i=1}^{n} \lambda_i^2/2} = \frac{3\alpha^2 - 5\alpha + 3}{\alpha^2} n + \frac{5\alpha - 5}{\alpha^2} n^2 + \frac{2}{\alpha^2} n^3$$

⇒ Moments of the determinant: [redacted]

$$\int_{\mathbb{R}^6} \prod_{i=1}^{6} \lambda_i^3 \prod_{i<j} |\lambda_i - \lambda_j|^{2/\alpha} \, e^{-\sum_{i=1}^{6} \lambda_i^2/2} =$$

$$= -75 \, \frac{25a^6 + 153a^5 + 472a^4 + 693a^3 + 610a^2 + 207a + 45}{a^9}$$

Symbolic MOPS applications

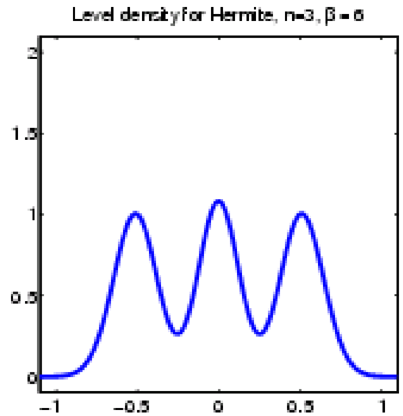
Applications: eigenvalue statistics

Examples for Hermite:

⇒ Level densities: [redacted]

$$\int_{\mathbf{R}^2} \prod_{i=1}^{2} |\lambda_i - x|^6 \prod_{i<j} |\lambda_i - \lambda_j|^6 \quad e^{-\sum_{i=1}^{2} \lambda_i^2/2} =$$

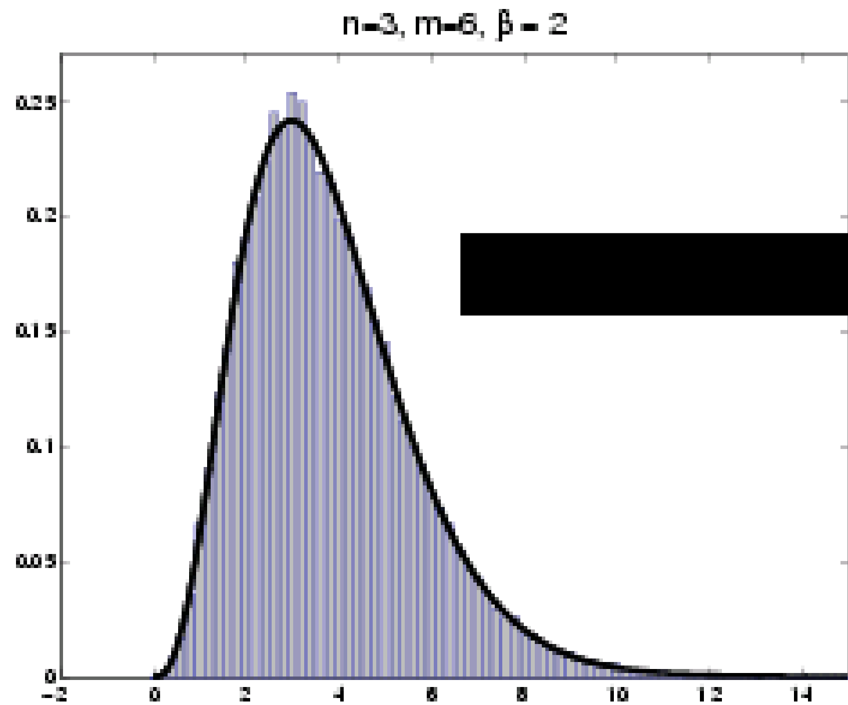
$$= \frac{3\sqrt{2}\, e^{-18x^2}}{2240\sqrt{\pi}} (80621568x^{12} + 26873856x^{10} + 8398080x^8 + 136080x^4 - 15120x^2 + 1015$$



The exact level density for the 3×3 Hermite ensemble with $\alpha = 1/3$

Smallest eigenvalue statistics

$$\rho(x) = x^9 \, e^{-3x/2} \; {}_2F_0^1(-3, 4; -2I_2/x) \; ;$$



[redacted]

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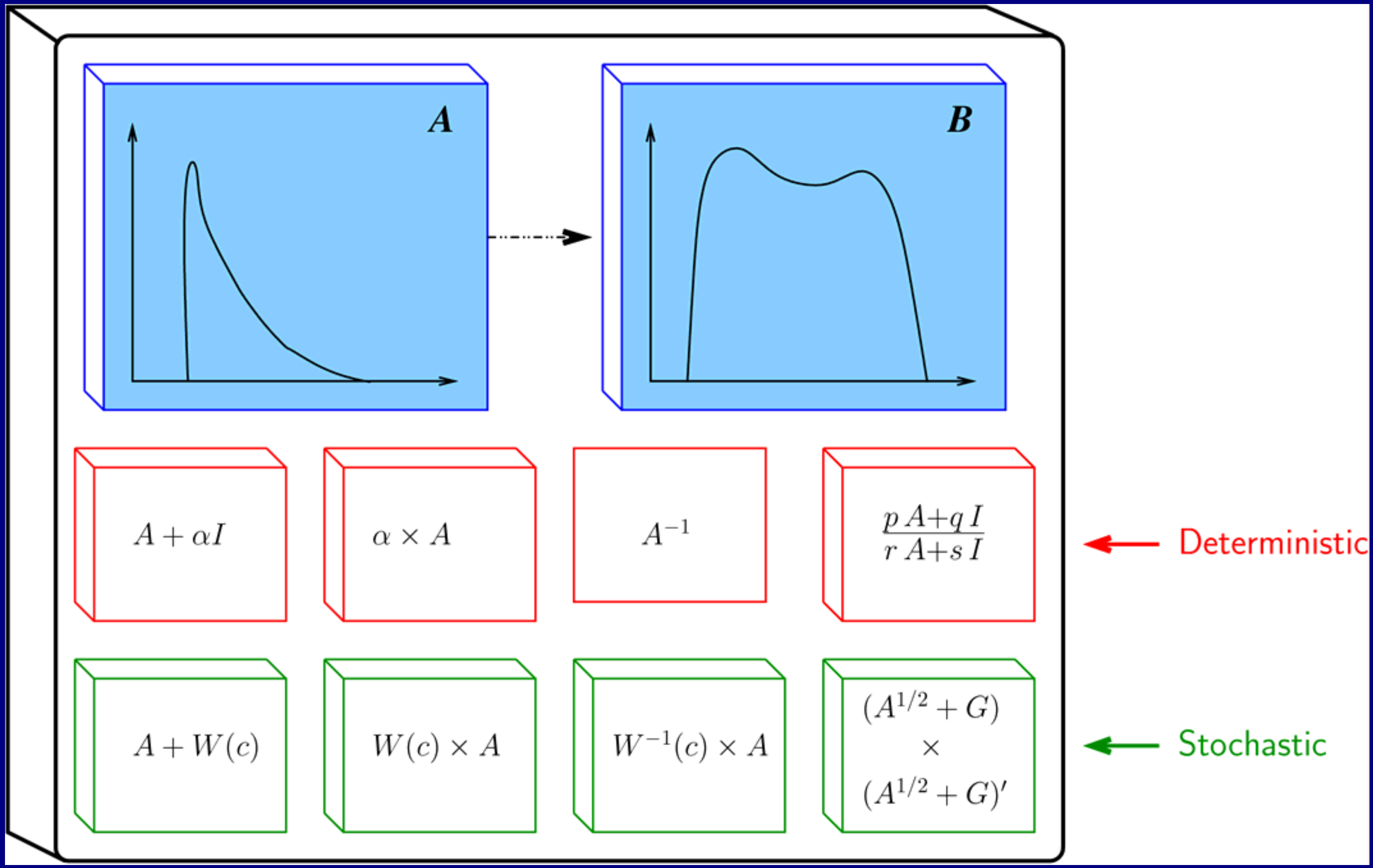
RM Tool – Raj!



Courtesy of the Polynomial Method

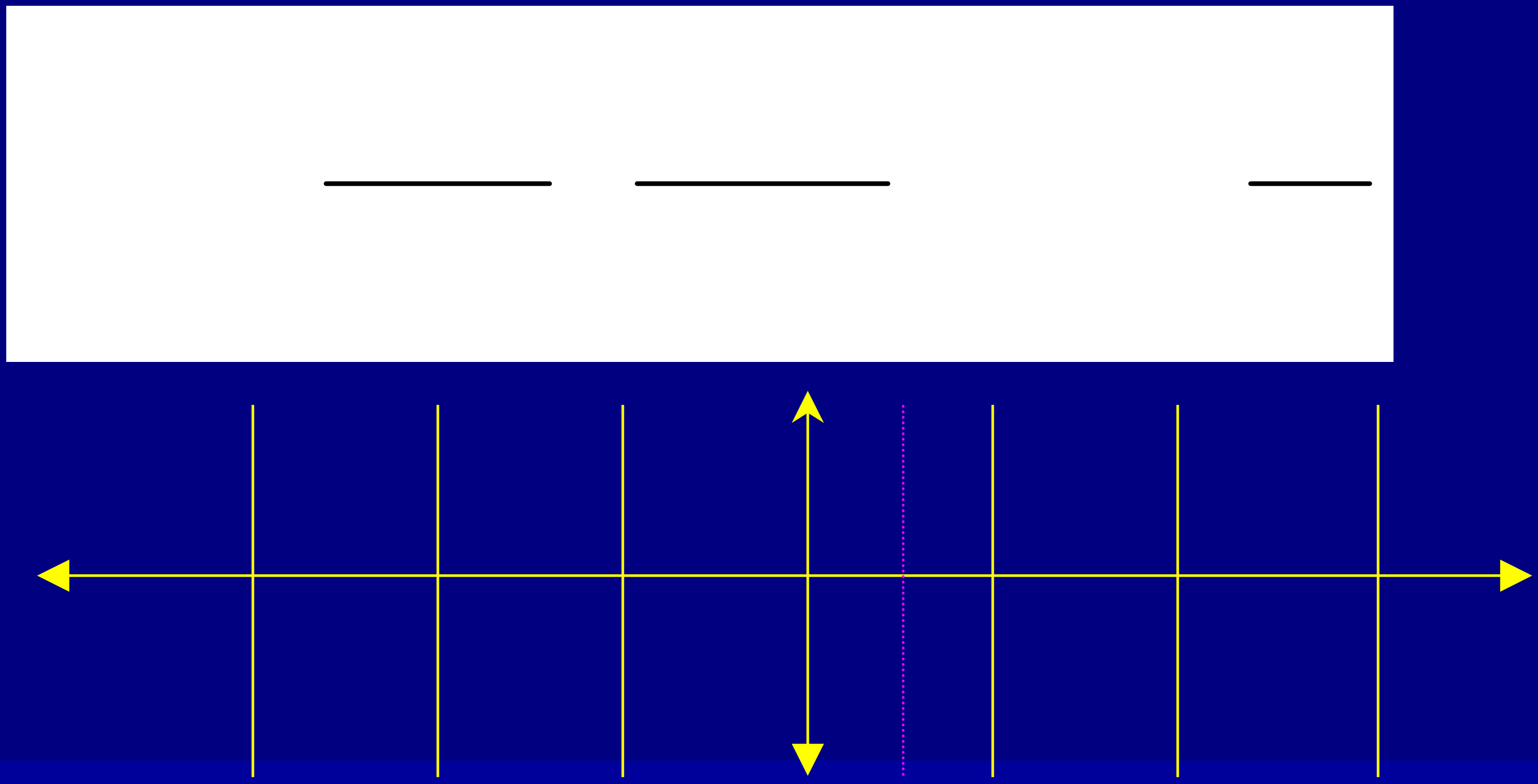
	Delta	Semi-circle Hermite	Sample Covariance Laguerre	Regular graphs Jacobi
$f(x)$	$f_r(x) = \delta(x-r)$	$f_r(x) = \frac{2}{\pi r^2}\sqrt{r^2-x^2}$	$f_r(x) = \frac{\sqrt{(x-r_1)(r_2-x)}}{2\pi x r}$ $r_1 = (1-\sqrt{r})^2$, $r_2 = (1+\sqrt{r})^2$, $r \le 1$	$f_r(x) = \frac{r(r-1)\sqrt{4-x^2}}{2\pi(r^2-(r-1)x^2)}$ $r \ge 2$
Moments	$m_k = r^k$	$m_{2k} = C_k(r/2)^{2k}$ C_k = "Catalan Number" $= \binom{2k}{k}/(k+1)$	$m_k = \sum_{j=0}^{k-1} \frac{r^j}{j+1}\binom{k}{j}\binom{k-1}{j}$ "Narayana Polynomials" ($m_k = C_k$ if $r = 1$)	$m_{2k} = \sum_{j=1}^{k} \binom{2k-j}{k}\frac{j}{2k-j}\left(\frac{r}{r-1}\right)^j$ ($m_{2k} = C_k$, if $r = \infty$)
Support	$x = r$	$x = [-r, r]$	$x = [r_1, r_2]$	$\|x\| \le 2$
Cauchy Transform	$m(z) = 1/(r-z)$	$m(z) = \frac{2}{r^2}(\sqrt{z^2-r^2}-z)$	$m(z) = \frac{1}{2} + \frac{1-r}{2z} + i\pi f_r(z)$	$m(z) = \text{P.V. } m(z) + i\pi f_r(z)$
Inverse Transform	$z(m) = -\frac{1}{m} + r$	$z(m) = -\frac{1}{m} - \frac{mr^2}{4}$	$z(m) = -\frac{1}{m} + \frac{r}{1+m}$	$z(m) = -\frac{1}{m} + \frac{r}{2m}\left(1-\sqrt{\frac{1+4m^2}{r-1}}\right)$
Symmetric Form	$mr - mz - 1 = 0$	$m^2r^2 + 4zm + 4 = 0$	$zm^2 + m(z+1-r) + 1 = 0$	$(r^2-(r-1)z^2)m^2 + (r-1)(r-2)zm + (r-1)^2 = 0$
Principal Value	—	$\text{P.V. } m(z) = -\frac{2}{r^2}z$	$\text{P.V. } m(z) = \frac{1+(1-r)/z}{2}$	$\text{P.V. } m(z) = \frac{(r-1)(r-2)z/2}{(r-1)z^2-r^2}$
R-Transform	$R(m) = r$	$R(m) = -mr^2/4$	$R(m) = r/(1+m)$	$R(m) = \frac{r}{2m}\left(1-\sqrt{1+\frac{4m^2}{r-1}}\right)$

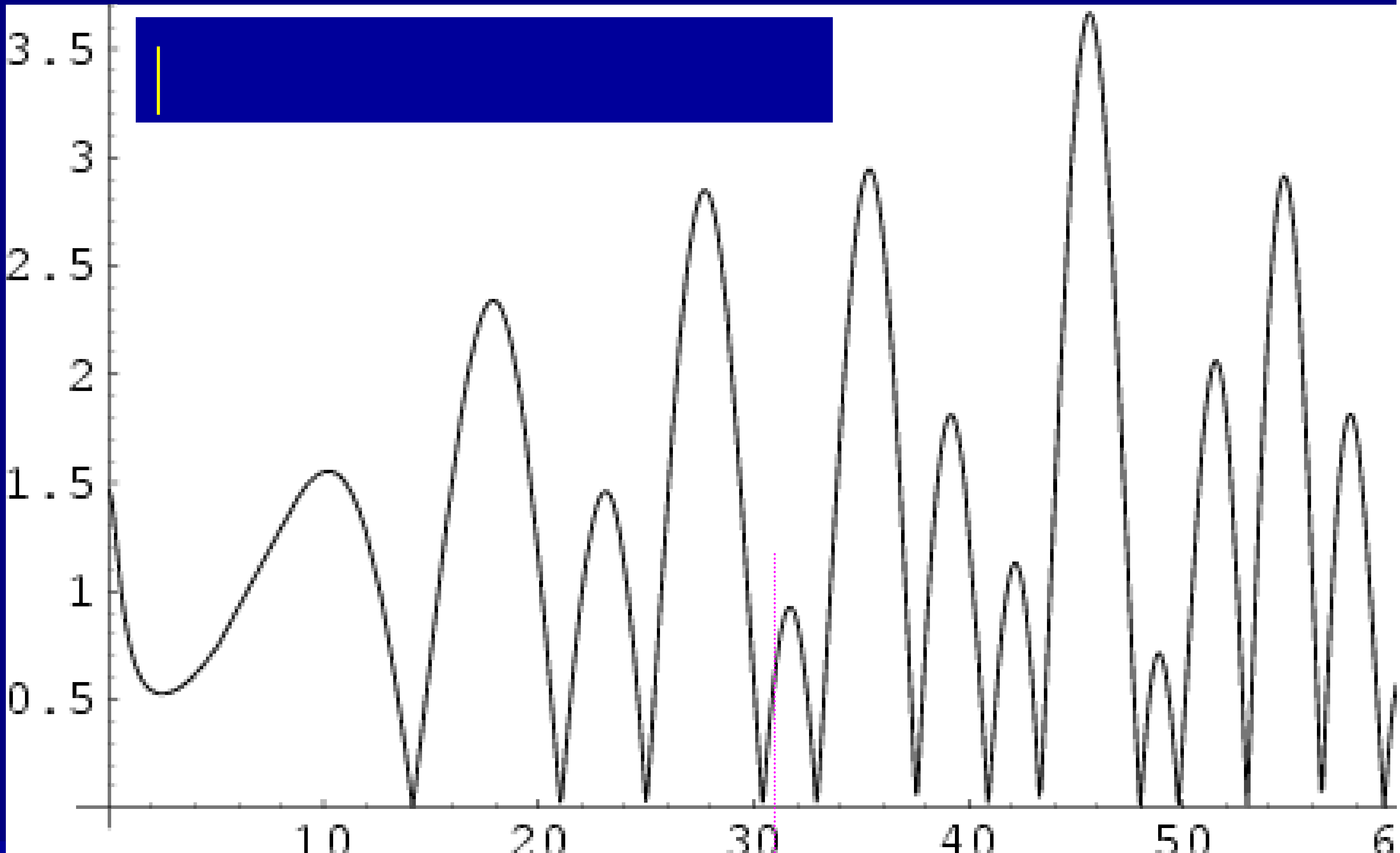
The Riemann Zeta Function

On the real line with x>1, for example

May be analytically extended to the complex plane, with singularity only at x=1.

The Riemann Hypothesis




3.5
3
2.5
2
1.5
1
0.5
10
20
30
40
50
6

Computation of Zeros

- Odlyzko's fantastic computation of 10^k+1 through 10^k+10,000 for k=12,21,22.

See http://www.research.att.com/~amo/zeta_tables/

Spacings behave like the eigenvalues of

A=randn(n)+i*randn(n); S=(A+A')/2;

Nearest Neighbor Spacings & Pairwise Correlation Functions

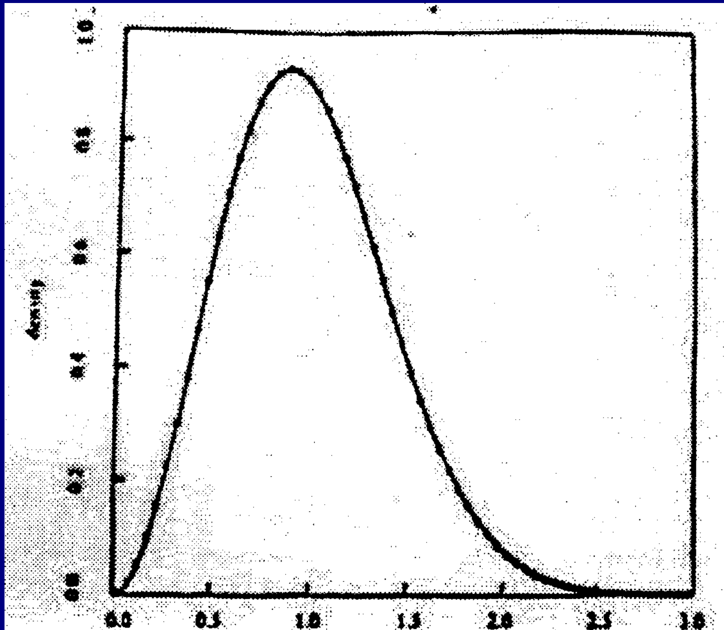


FIGURE 1
Nearest neighbor spacings among 70 million zeroes beyond the 10^{20}-th zero of zeta, versus $\mu_1(GUE)$

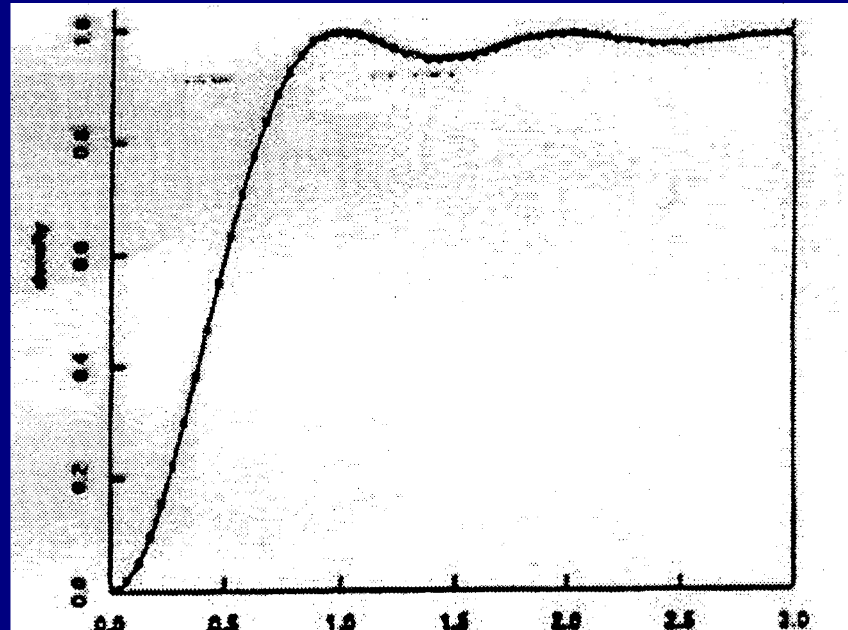


FIGURE 2
Pair correlation for zeros of zeta based on 8×10^6 zeros near the 10^{20}-th zero, versus the GUE conjectured density $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$.

Painlevé Equations

$$
\begin{aligned}
\text{I)} \qquad y'' &= 6y^2 + t, \\
\text{II)} \qquad y'' &= 2y^3 + ty + \alpha, \\
\text{III)} \qquad y'' &= \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \\
\text{IV)} \qquad y'' &= \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{V)} \qquad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) \\
&\qquad + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}, \\
\text{VI)} \qquad y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y'
\end{aligned}
$$

$$
+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(y-t)^2}\right]
$$

Spacings

- Take a large collection of consecutive zeros/eigenvalues.
- Normalize so that average spacing = 1.
- Spacing Function = Histogram of consecutive differences (the (k+1)st – the kth)
- Pairwise Correlation Function = Histogram of all possible differences (the kth – the jth)
- Conjecture: These functions are the same for random matrices and Riemann zeta

Tools

- Motivation: A condition number problem
- Jack & Hypergeometric of Matrix Argument
- MOPS: Ioana Dumitriu's talk
- The Polynomial Method
- The tridiagonal numerical 10^9 trick

Everyone's Favorite Tridiagonal

$$\frac{1}{n^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}$$

$$\boxed{\frac{d^2}{dx^2}}$$

Everyone's Favorite Tridiagonal

$$\frac{1}{n^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} + \frac{1}{(\beta n)^{1/2}}\begin{pmatrix} G & & & \\ & G & & \\ & & \ddots & \\ & & & G \end{pmatrix}$$

$$\boxed{\frac{d^2}{dx^2}} + \boxed{\frac{dW}{\beta^{1/2}}}$$

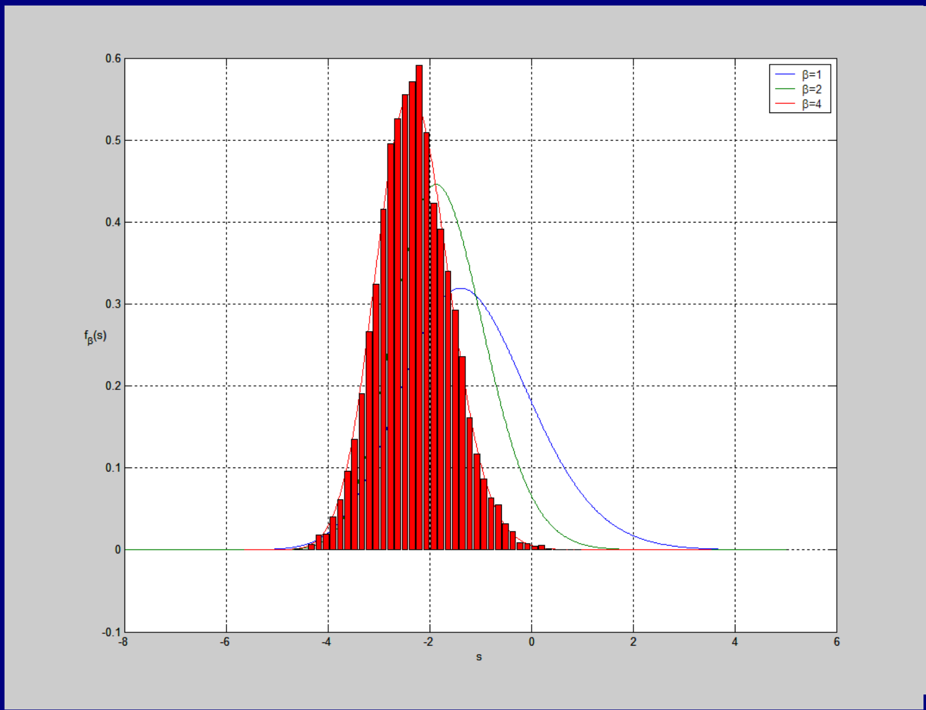
Stochastic Operator Limit

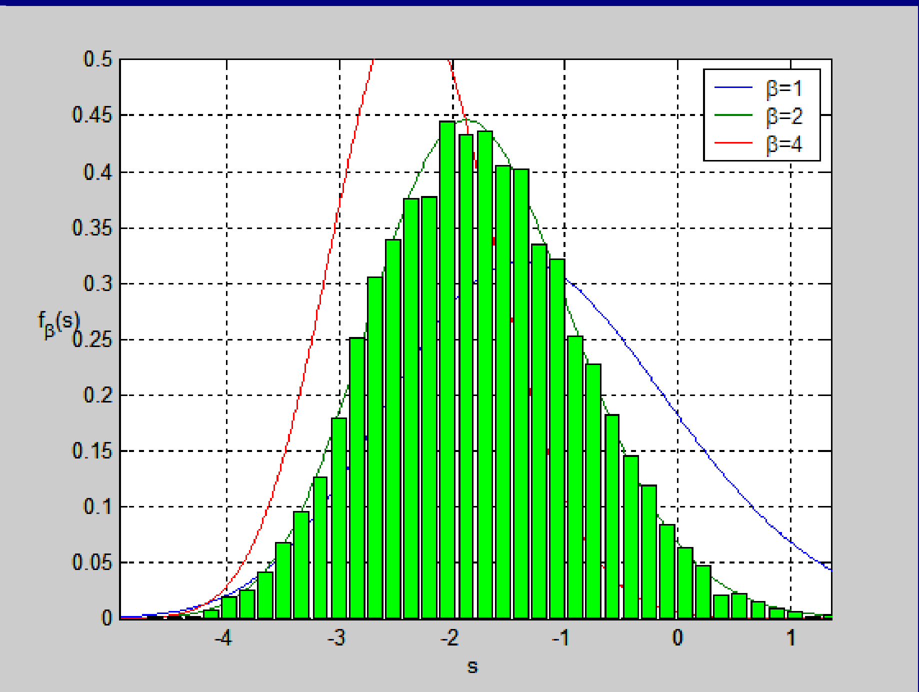
$$\frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}} dW ,$$

$$H_n^\beta \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix},$$

$$H_n^\beta \approx H_n^\infty + \frac{2}{\sqrt{\beta}} G_n ,$$

Largest Eigenvalue Plots





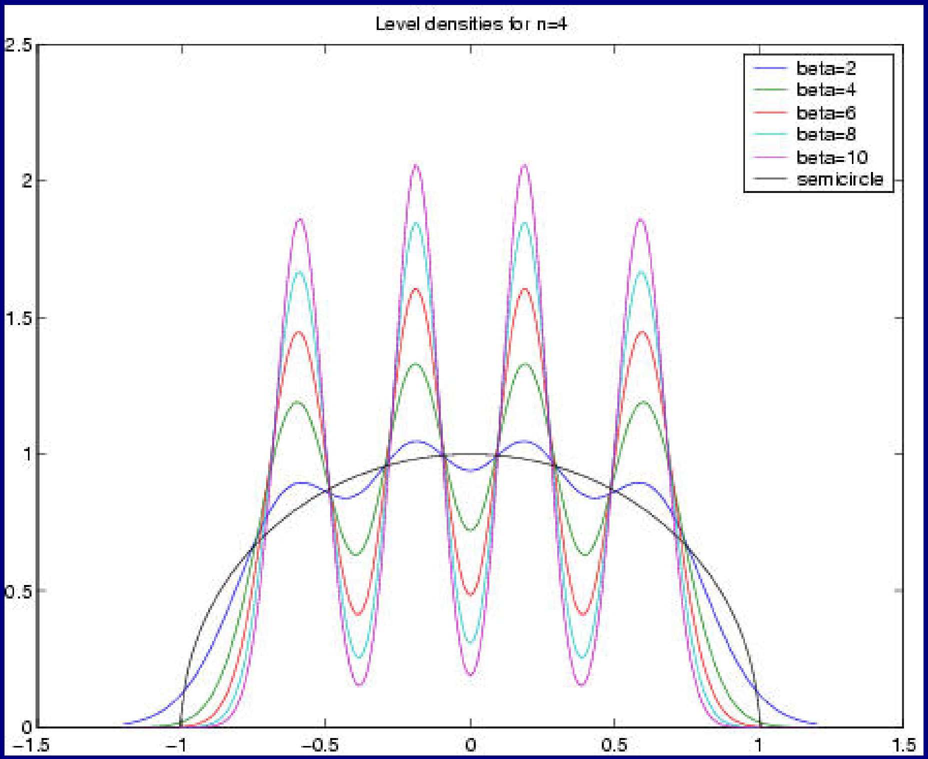
MATLAB

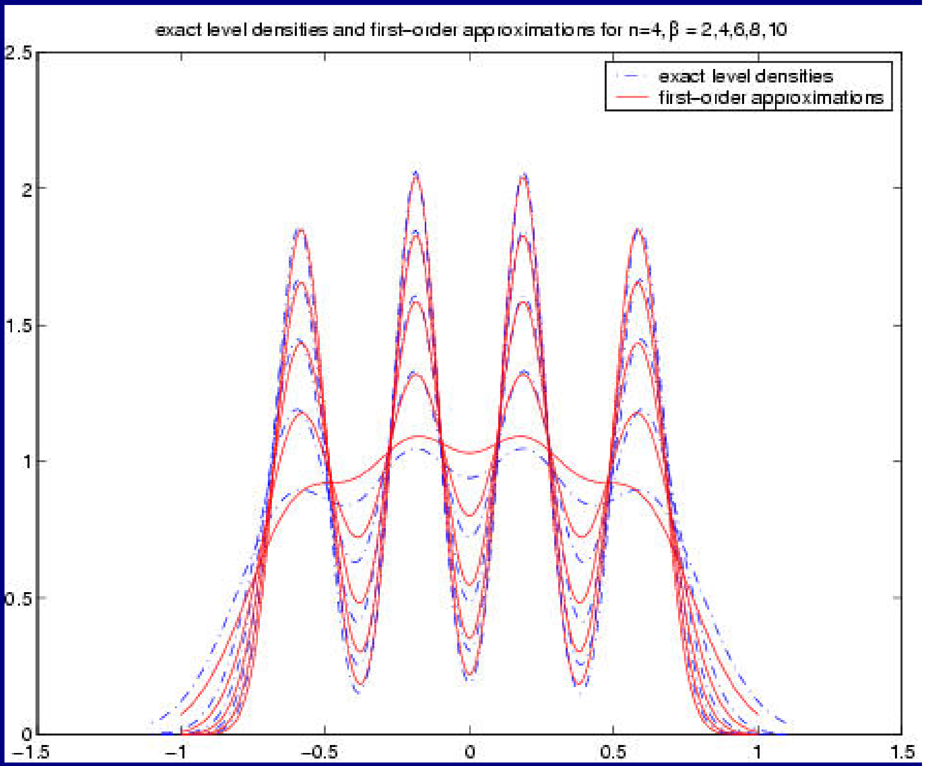
```
beta=1; n=1e9; opts.disp=0;opts.issym=1;
alpha=10; k=round(alpha*n^(1/3));  % cutoff parameters
d=sqrt(chi2rnd( beta*(n:-1:(n-k-1))))';
H=spdiags( d,1,k,k)+spdiags(randn(k,1),0,k,k);
H=(H+H')/sqrt(4*n*beta);
eigs(H,1,1,opts)
```

Tricks to get $O(n^9)$ speedup

- Sparse matrix storage (Only O(n) storage is used)
- Tridiagonal Ensemble Formulas (Any beta is available due to the tridiagonal ensemble)
- The Lanczos Algorithm for Eigenvalue Computation (This allows the computation of the extreme eigenvalue faster than typical general purpose eigensolvers.)
- The shift-and-invert accelerator to Lanczos and Arnoldi (Since we know the eigenvalues are near 1, we can accelerate the convergence of the largest eigenvalue)
- The ARPACK software package as made available seamlessly in MATLAB (The Arnoldi package contains state of the art data structures and numerical choices.)
- The observation that if $k = 10n^{1/3}$, then the largest eigenvalue is determined numerically by the top $k \times k$ segment of n. (This is an interesting mathematical statement related to the decay of the Airy function.)

Level Densities





Open Problems

The distribution for general beta
Seems to be governed by a convection-diffusion equation

Random matrix tools!