on the **geometry of QR**

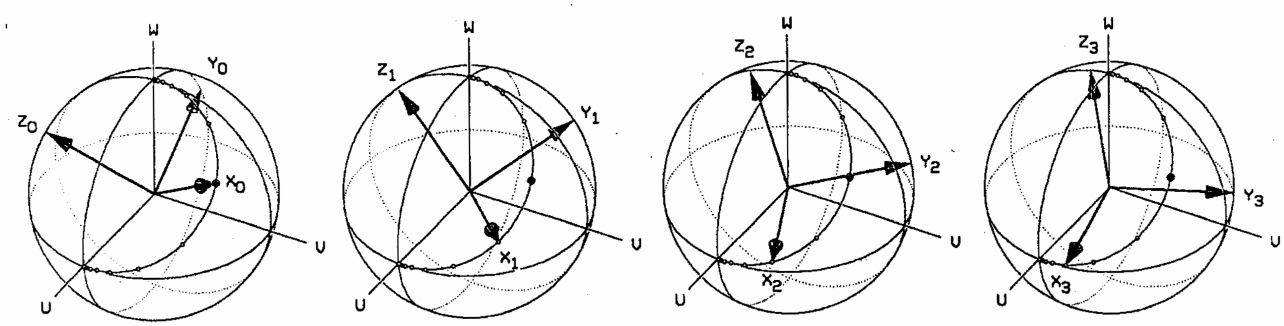
18.330 Handout cribbed from Fri 07 May 04

18.335 Problem Set 9 Due: Friday, Dec. 8, 1989

Preamble:

As shown in class (and again hinted on a handout) the **basic QR algorithm** $Q_kR_k = A_k$, $A_{k+1} = R_kQ_k \equiv Q_k^TA_kQ_k$ applied to any symmetric 3×3 matrix $A \equiv A_0$ is logically equivalent either to the **simultaneous iteration of the three vectors** $x_k = Ax_{k-1}$, $y_k = Ay_{k-1}$, $z_k = Az_{k-1}$ after starting from the humble trio $x_0 = [1,0,0]^T$, $y_0 = [0,1,0]^T$, $z_0 = [0,0,1]^T$, or else just to **raising the given matrix A to the k-th power** and then reading out its column vectors.

The only catch is that one must convert this evolving trio into a mutually orthogonal triplet X_k = const • x_k , Y_k = portion of y_k orthogonal to x_k , and Z_k = orthogonal to both x_k and y_k , before using the same to estimate the eigenvalues of A via **Rayleigh quotients** like $RQ_Y(k) = (Y_k^TAY_k)/(Y_k^TY_k)$.



To repeat, for this purpose it does not matter whether these "refined" X_k, Y_k, Z_k remain of unit length, but their orthogonality is essential. The QR algorithm very sensibly re-orthogonalizes at every step, but in principle (as here) we could do it only when actually needed at the end of the iteration.

Now please explore these ideas via the following single long problem devoted to just the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \text{with eigenvectors} \quad U = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and eigenvalues 4, 2, 1 for our mutual convenience.

CONTINUES ⇒

(26) (a) Supply and check explicit formulas for x_k, y_k, z_k, and thence also for X_k, Y_k, Z_k ($= x_k \times y_k$ here, I do believe!).

(b) Confirm analytically that $RQ_X(k) = (2^{2k+1}+1)/(2^{2k-1}+1)$.

(c) Derive a similarly compact formula for $RQ_Z(k)$ ... and check it, too, against the output of your QR algorithm.

(d) Show from first principles that the <u>sum</u> of the Rayleigh quotients referring to any three mutually orthogonal trial vectors must always equal 7 for the present matrix.

(e) Then derive (and check) a fairly tidy formula even for $RQ_Y(k)$.

(f) Show that on the direction sphere anchored on the **U**, **V**, **W** triplet of eigenvectors, our "middle" vector Y_k hops exactly along the arc of a <u>great circle</u> ... and travels 90^o altogether.

(g) Check analogously whether even the landing spots of our main iterated vector X_k lie on a <u>small circle</u> on that direction sphere. If not, could it have been moved onto one via a change of starting vector?

(h) What precise sense, if any, can you make of these iterated vectors for <u>negative</u> values of k ?

And last but not least,

(i) LIGHTLY REPEAT these main themes for the closely related matrix

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix},$$

which happens to have exactly the same eigenvectors as above, but eigenvalues 4, -2, 1 to make life a little more interesting.