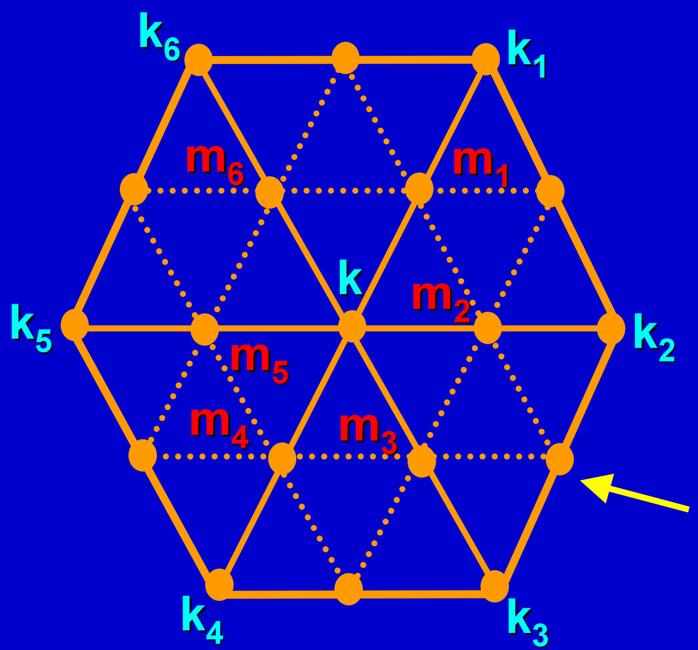
Course 18.327 and 1.130
Wavelets and Filter Banks

Wavelets and subdivision: nonuniform grids; multiresolution for triangular meshes; representation and compression of surfaces.

Wavelets on Surfaces in R^3

Construction by Schröder and Sweldens

- uses lifting
- scaling functions are interpolating in most straightforward case
- typically work with triangular mesh generated by subdivision



$K(j) = \{k, k_1, k_2, k_3, k_4, k_5, k_6, \ldots\}$

$M(j) = \{m_1, m_2, m_3, m_4, m_5, m_6, \ldots\}$

mesh describing surface S

Notation:

K(j) = all vertices at resolution j

K(j + 1) = all vertices at resolution j + 1

M(j) = vertices obtained by subdividing the resolution j mesh to produce the resolution j + 1 mesh

So

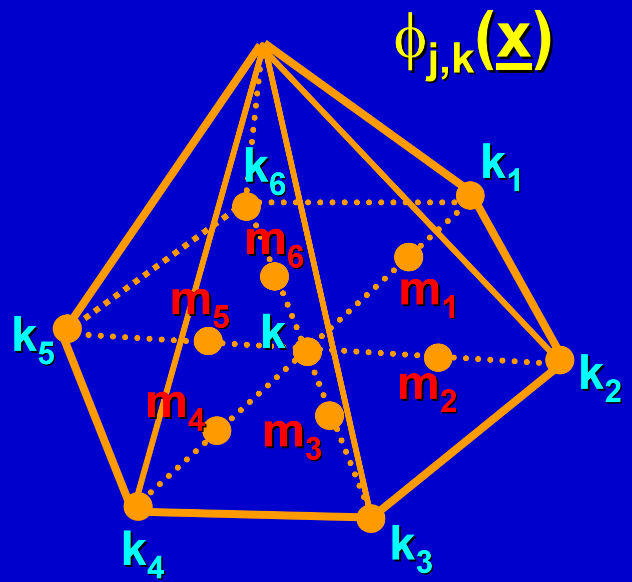
$$K(j+1) = K(j) \setminus M(j)$$

Interpolating property means that scalings functions satisfy

$$\phi_{j,k}(\underline{x}) = \begin{cases} 1 \text{ if } \underline{x} = \underline{x}_k & k \in K(j) \\ 0 \text{ if } \underline{x} = \underline{x}_{k'} & k' \in K(j),\ k' \neq k \end{cases}$$

$\underline{x}$ = position vector of a point on S.

Simple interpolating scaling function: hat function



Scaling functions at level j are all located at vertices in K(j)

Refinement equation

$$\phi_{j,k}(\underline{x}) = \phi_{j+1,k}(\underline{x}) + \tfrac{1}{2}\sum_{m=m_1}^{m_6} \phi_{j+1,m}(\underline{x})$$

In general, interpolating scaling functions will satisfy a refinement equation of the form

$$\boxed{\phi_{j,k}(\underline{x}) = \phi_{j+1,k}(\underline{x}) + \sum_{m\in n(j,k)} h_0^j[k,m]\,\phi_{j+1,m}(\underline{x})}$$

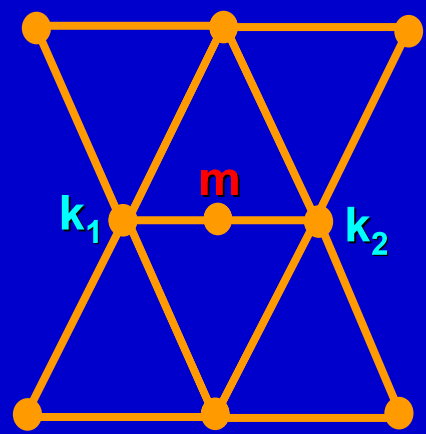
n(j,k) = vertices in the neighborhood of vertex k that contribute to the refinement equation. Because of interpolating property, n(j,k) can only consist of vertices in M(j).

How to construct the wavelet?

Start with

$w_{j,m}(\underline{x}) = \phi_{j+1,m}(\underline{x})$ Wavelets at level j are all located at vertices in M(j)

Then use the lifting idea to impose vanishing moment.



Consider a wavelet of the form

$$w_{j,m}(\underline{x}) = \phi_{j+1,m}(\underline{x}) - \alpha_1 \phi_{j,k_1}(\underline{x}) - \alpha_2 \phi_{j,k_2}(\underline{x})$$

For the zero^th moment to vanish

$$0 = I_{j+1,m} - \alpha_1 I_{j,k_1} - \alpha_2 I_{j,k_2}$$

where

$$I_{j,k} = \int_s \phi_{j,k}(\underline{x}) dS$$

To satisfy vanishing moment condition, choose

$$\alpha_i = I_{j+1,m}/2I_{j,k_i} \qquad i = 1, 2$$

So the wavelet equation can be written as

$$w_{j,m}(\underline{x}) = \phi_{j+i,m}(\underline{x}) - \sum_{k \in A(j,m)} h_1^j[k,m]\phi_{j,k}(\underline{x})$$

with

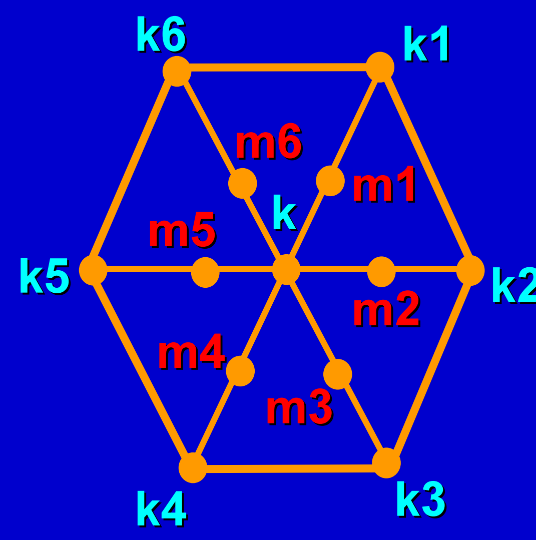
$A(j,m)$ = two immediate neighbors in $K(j)$

$$h_1^j[k,m] = I_{j+1,m}/2I_{j,k}$$

Wavelets on Surfaces in R^3

Synthesis scaling function

$$\phi_{j,k}(\underline{x}) = \phi_{j+1,k}(\underline{x}) + \sum_{m \in n(j,k)} h_0^j[k,m]\phi_{j+1,m}(x)$$



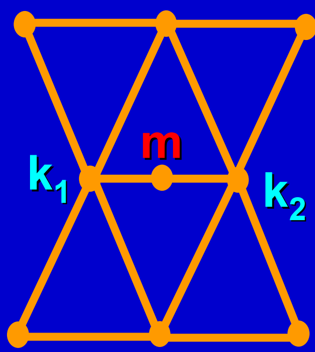
Linear interpolating functions:

$$h_0^j[k,m] = \begin{cases} \tfrac{1}{2} & m \in n(j,k) \\ 0 & \text{otherwise} \end{cases}$$

$$n(j,k) = \{m_1, m_2, m_3, m_4, m_5, m_6\}$$

Synthesis wavelet

$$w_{j,m}(\underline{x}) = \phi_{j+1,m}(\underline{x}) - \sum_{k \in A(j,m)} h_1^j[k,m]\,\phi_{j,k}(\underline{x})$$



$$A(j,m) = \{k_1, k_2\}$$

What are the analysis functions?

Use alternating signs condition to get analysis filters, e.g. 1D interpolating filter

If $F_0(z) = \frac{1}{16}\{-z^3 + 0 \bullet z^2 + 9z + 16 + 9z^{-1} + 0 \bullet z^{-2} - z^{-3}\}$

then $H_1(z) = F_0(-z) = \frac{1}{16}\{z^3 + 0.z^2 - 9z + 16 - 9z^{-1} + 0.z^{-2} + z^{-3}$

$\Rightarrow$Change signs of all coefficients except center

So the analysis functions turn out to be

$$\tilde{\phi}_{j,k}(\underline{x}) = \tilde{\phi}_{j+1,k}(\underline{x}) + \sum_{m \in a(j,k)} h_1^j[k,m]\tilde{w}_{j,m}(\underline{x}) \quad a(j,k)=\{m:k \in A(j,m)\}$$

$$\tilde{w}_{j,m}(x) = \tilde{\phi}_{j+1,m}(\underline{x}) - \sum_{k \in N(j,m)} h_0^j[k,m]\tilde{\phi}_{j+1,k}(\underline{x}) \quad N(j,m)=\{k:m \in n(j,k)\}$$

Exercise: verify that $\phi_{j,k}(\underline{x})$, $w_{j,m}(\underline{x})$, $\tilde{\phi}_{j,k}(\underline{x})$, $\tilde{w}_{j,m}(\underline{x})$ are biorthogonal.

Equations for the DWT:

Analysis (from analysis wavelet, refinement equations)

$$d^j[m] = c^{j+1}[m] - \sum_{k \in N(j,m)} h_0^j[k,m] c^{j+1}[k] \qquad \text{predict}$$

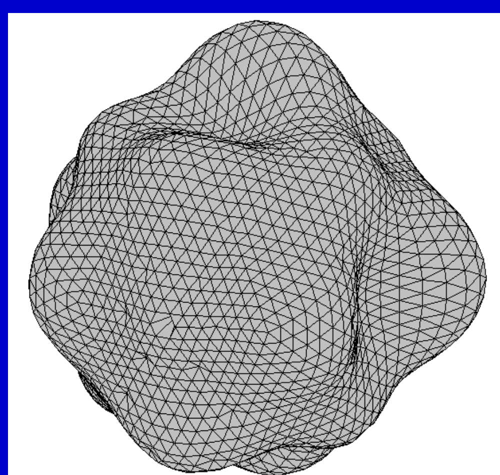
$$c^j[k] = c^{j+1}[k] + \sum_{m \in a(j,k)} h_1^j[k,m]\, d^j[m] \qquad \text{update}$$

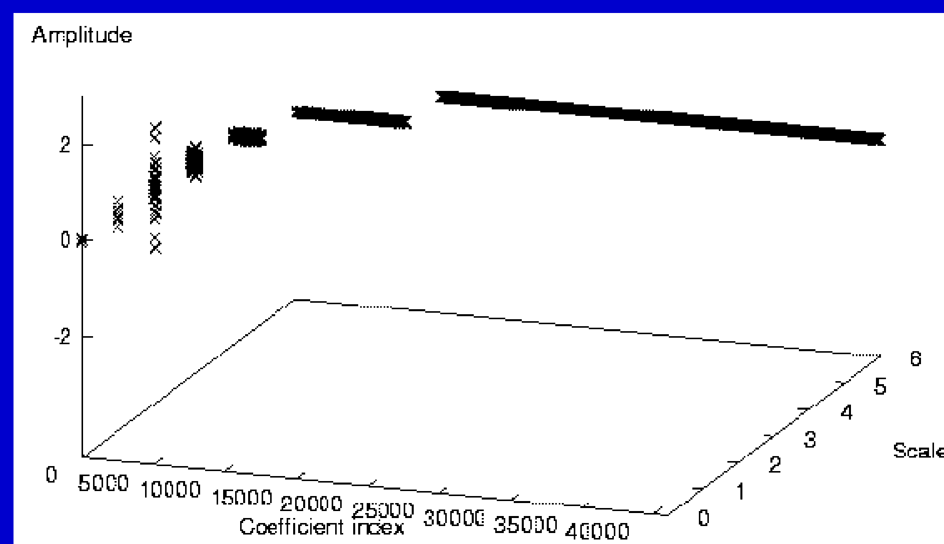
Synthesis (invert the lifting operations)

$$c^{j+1}[k] = c^j[k] - \sum_{m \in a(j,k)} h_1^j[k,m] d^j[m]$$

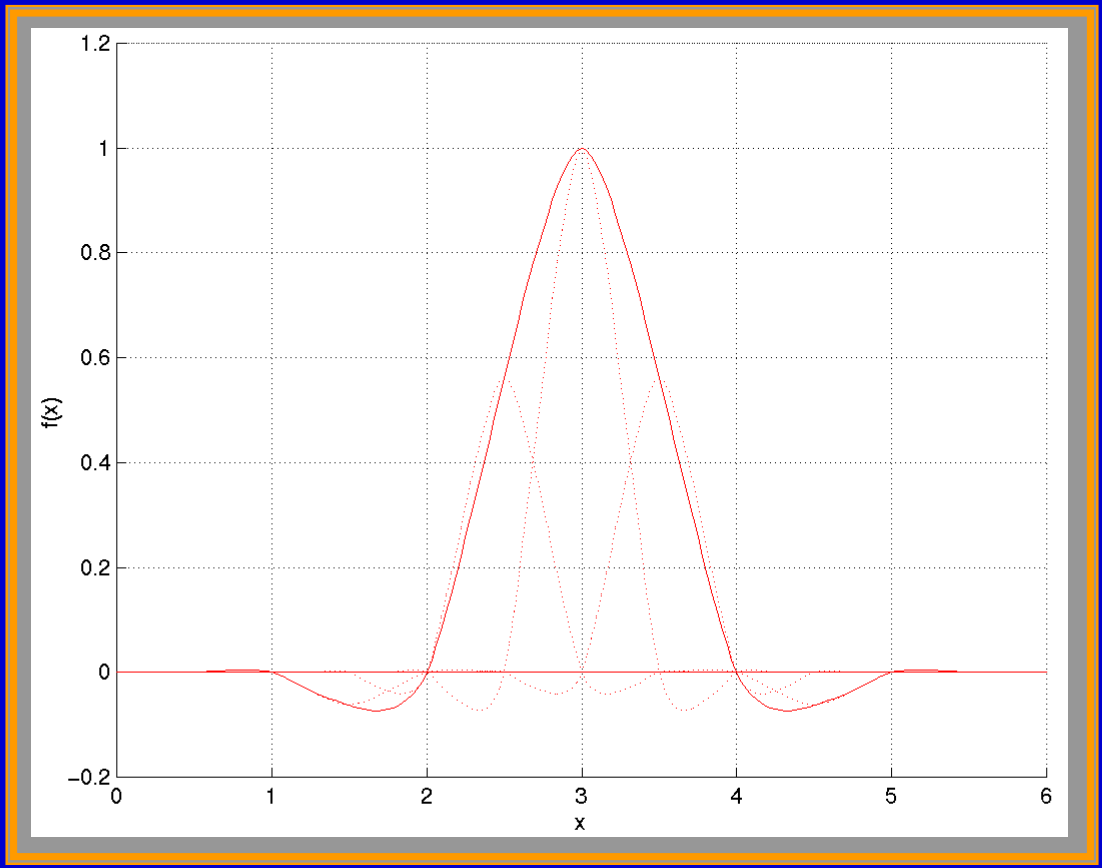
$$c^{j+1}[m] = d^j[m] + \sum_{k \in N(j,m)} h_0^j[k,m] c^{j+1}[k]$$

e.g.



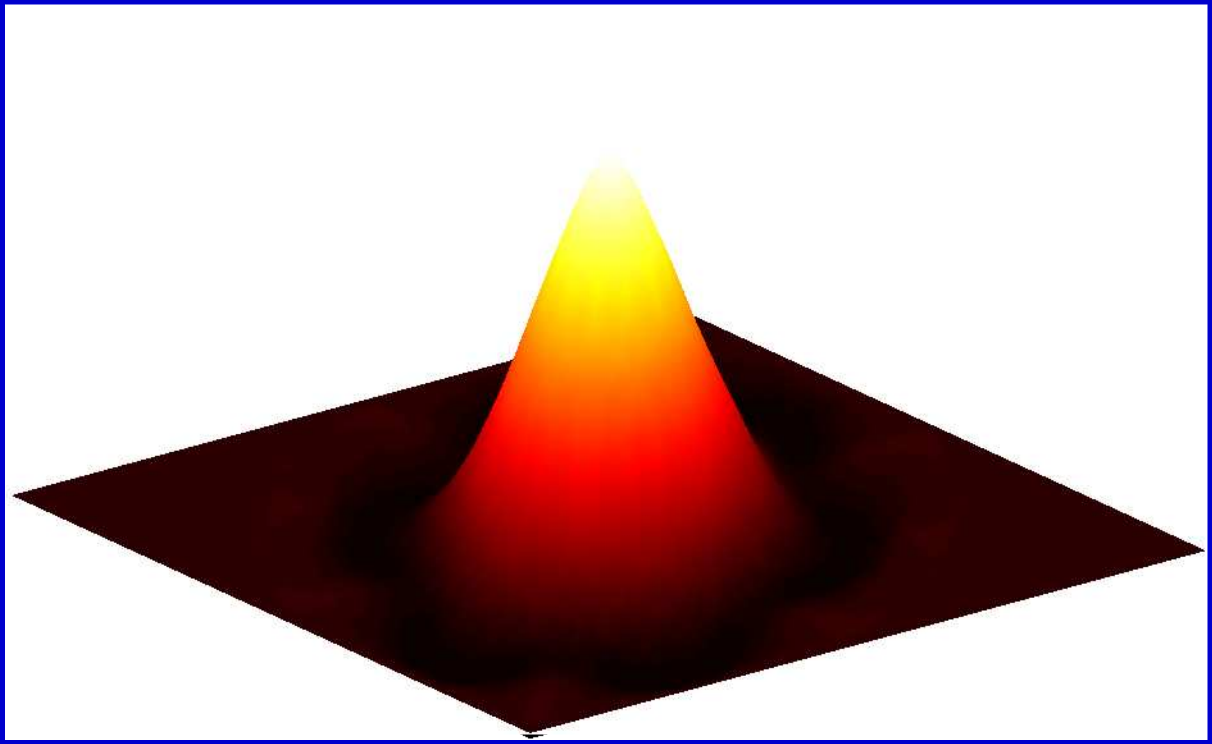


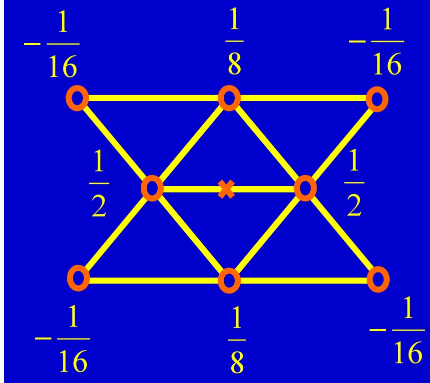
Cubic Interpolating Scaling Function



$$\phi(x) = \sum_k h_0[k]\phi(2x - k) \quad h_0[k] = \left\{-\frac{1}{16}, 0, -\frac{9}{16}, 1, -\frac{9}{16}, 0, -\frac{1}{16}\right\}$$

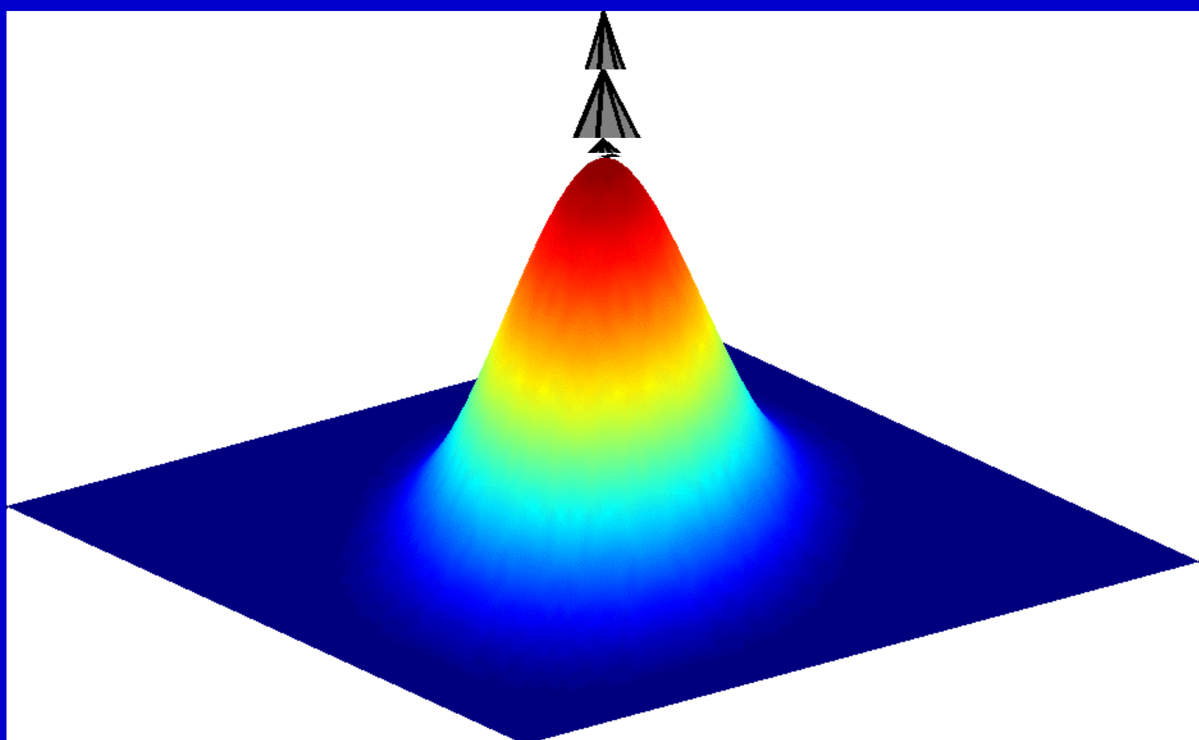
Butterfly Subdivision

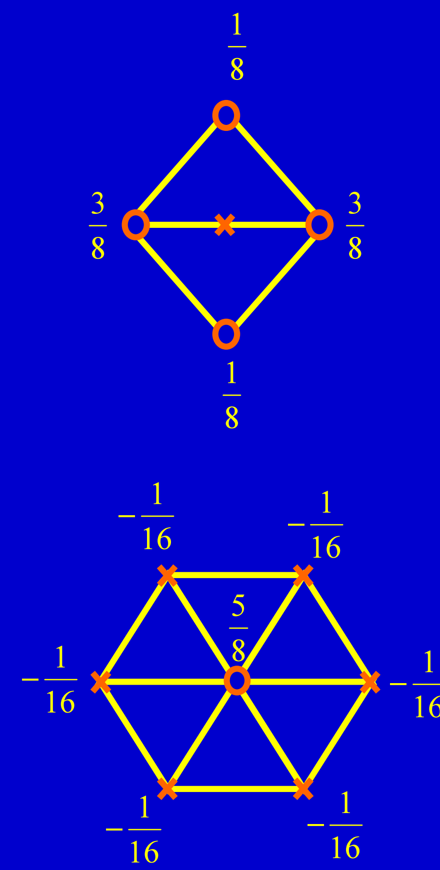




Also an interpolating function

Loop SubdivisionS





Not an interpolating function

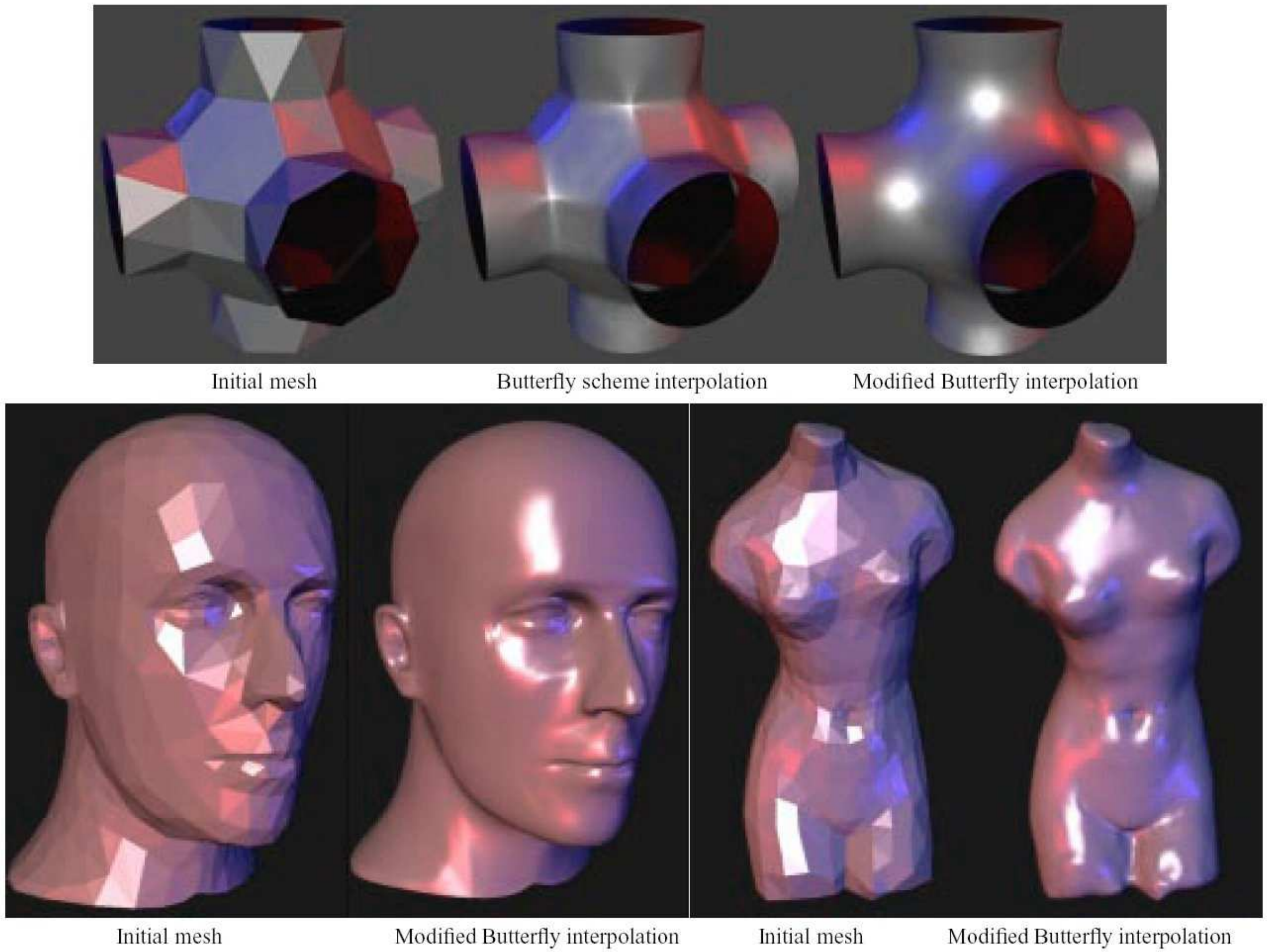


Figure 4 : Top row: pipe joint. Note the difference between Butterfly and Modified Butterfly. Lower left: mannequin head. Lower right: torso.

From: Zorin, Schroder and Sweldens, Interpolating subdivision for meshes with arbitrary topology, proceedings SIGGRAPH 1996.