

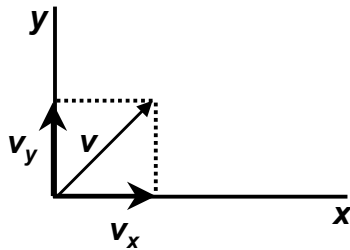
## Course 18.327 and 1.130 Wavelets and Filter Banks

Orthogonal wavelet bases: connection to orthogonal filters; orthogonality in the frequency domain. Biorthogonal wavelet bases.

### Orthogonal Wavelets

2D Vector Space:

Basis vectors are  $i, j$   $\rightarrow$  orthonormal basis



$v_x$  and  $v_y$  are the projections of  $v$  onto the  $x$  and  $y$  axes:

$$v_x = \langle v, i \rangle i$$

$$v_y = \langle v, j \rangle j$$

$$\langle v, i \rangle = [v_x \ v_y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_x \quad \text{Inner Product}$$

**Orthogonal multiresolution spaces:**

$V_j$  has an orthonormal basis  $\underbrace{\{2^{j/2} \phi(2^j t - k) : -\infty \leq k \leq \infty\}}_{\phi_{j,k}(t)}$

$W_j$  has an orthonormal basis  $\underbrace{\{2^{j/2} w(2^j t - k) : -\infty \leq k \leq \infty\}}_{w_{j,k}(t)}$

3

**Orthonormal means**

$$\langle \phi_{j,k}(t), \phi_{j,l}(t) \rangle = \int_{-\infty}^{\infty} 2^{j/2} \phi(2^j t - k) 2^{j/2} \phi(2^j t - l) dt = \delta[k - l]$$

$$\langle w_{j,k}(t), w_{j,l}(t) \rangle = \int_{-\infty}^{\infty} 2^{j/2} w(2^j t - k) 2^{j/2} w(2^j t - l) dt = \delta[k - l]$$

**For orthogonal multiresolution spaces, we have**

$$V_j \perp W_j$$

**So**

$$\langle \phi_{j,k}(t), w_{j,l}(t) \rangle = 0$$

4

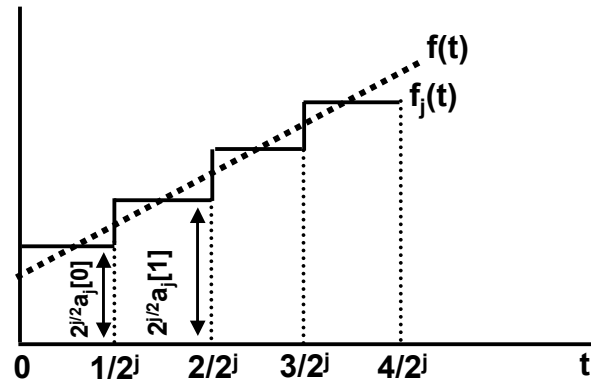
Projection of an  $L^2$  function,  $f(t)$ , onto  $V_j$ :

$$f_j(t) = \sum_k a_j[k] \phi_{j,k}(t)$$

with

$$a_j[k] = \langle f(t), \phi_{j,k}(t) \rangle$$

Haar example



5

Projection of  $f(t)$  onto  $W_j$ :

$$g_j(t) = \sum_k b_j[k] w_{j,k}(t)$$

with

$$b_j[k] = \langle f(t), w_{j,k}(t) \rangle$$

6

## Biorthogonal Wavelet Bases

Two scaling functions and two wavelets:

Synthesis:

$$\phi(t) = 2 \sum_k f_0[k] \phi(2t - k)$$

$$w(t) = 2 \sum_k f_1[k] \phi(2t - k)$$

Analysis:

$$\tilde{\phi}(t) = 2 \sum_k h_0[-k] \tilde{\phi}(2t - k)$$

$$\tilde{w}(t) = 2 \sum_k h_1[-k] \tilde{\phi}(2t - k)$$

7

Two sets of multiresolution spaces:

$$\{0\} \subset \dots V_0 \subset V_1 \subset \dots \subset V_j \subset \dots \subset L^2(\square)$$

$$\{0\} \subset \dots \tilde{V}_0 \subset \tilde{V}_1 \subset \dots \subset \tilde{V}_j \subset \dots \subset L^2(\square)$$

$$V_j + W_j = V_{j+1}$$

$$\tilde{V}_j + \tilde{W}_j = \tilde{V}_{j+1}$$

Spaces are orthogonal w.r.t. each other i.e.

$$V_j \perp \tilde{W}_j \quad \tilde{V}_j \perp W_j$$

$V_0$  has a basis  $\{\phi(t - k) : -\infty < k < \infty\}$

$\tilde{V}_0$  has a basis  $\{\tilde{\phi}(t - k) : -\infty < k < \infty\}$

$W_0$  has a basis  $\{w(t - k) : -\infty < k < \infty\}$

$\tilde{W}_0$  has a basis  $\{\tilde{w}(t - k) : -\infty < k < \infty\}$

8

**Bases are orthogonal w.r.t. each other i.e.**

$$\int \phi(t) \tilde{\phi}(t-k) dt = \delta[k] \quad \int \phi(t) \tilde{w}(t-k) dt = 0$$

$$\int w(t) \tilde{\phi}(t-k) dt = 0 \quad \int w(t) \tilde{w}(t-k) dt = \delta[k]$$

**Equivalent to perfect reconstruction conditions on filters**

**Representation of functions in a biorthogonal basis:**

$$f(t) = \sum_k c_k \phi(t-k) + \sum_{j=0}^{\infty} \sum_k d_{j,k} 2^{j/2} w(2^j t - k)$$

$$c_k = \int f(t) \tilde{\phi}(t-k) dt$$

$$d_{j,k} = 2^{j/2} \int f(t) \tilde{w}(2^j t - k) dt$$

9

**Similarly, we can represent  $f(t)$  in the dual basis**

$$f(t) = \sum_k \tilde{c}_k \tilde{\phi}(t-k) + \sum_{j=0}^{\infty} \sum_k \tilde{d}_{j,k} 2^{j/2} \tilde{w}(2^j t - k)$$

$$\tilde{c}_k = \int f(t) \phi(t-k) dt$$

$$\tilde{d}_{j,k} = 2^{j/2} \int f(t) w(2^j t - k) dt$$

**Note: When  $f_0[k] = h_0[-k]$  and  $f_1[k] = h_1[-k]$ , we have**

$$\phi(t) = \tilde{\phi}(t) \Rightarrow V_j = \tilde{V}_j$$

$$w(t) = \tilde{w}(t) \Rightarrow W_j = \tilde{W}_j$$

**i.e. we have orthogonal wavelets!**

10

**Connection between orthogonal wavelets and orthogonal filters**

**Start with the orthonormality requirement on scaling functions**

$$\delta[n] = \int_{-\infty}^{\infty} \phi(t) \phi(t - n) dt$$

**And then change scale using the refinement equation:**

$$\begin{aligned} \delta[n] &= \int_{-\infty}^{\infty} 2 \sum_k h_0[k] \phi(2t - k) 2 \sum_l h_0[l] \phi(2(t - n) - l) dt \\ &= 4 \sum_k h_0[k] \sum_l h_0[l] \int_{-\infty}^{\infty} \phi(\tau) \phi(\tau + k - 2n - l) d\tau/2 \\ &= 2 \sum_k h_0[k] \sum_l h_0[l] \delta[-k + 2n + l] \end{aligned}$$

$$\delta[n] = 2 \sum_k h_0[k] h_0[k - 2n]$$

**This is the “double shift” orthogonality condition that we previously encountered for orthogonal filters.**

11

**So orthogonal filters are necessary for orthogonal wavelets. Are they sufficient?**

**Consider the cascade algorithm**

$$\phi^{(i+1)}(t) = 2 \sum_{k=0}^N h_0[k] \phi^{(i)}(2t - k)$$

**If the filters are orthogonal and**

$$\int_{-\infty}^{\infty} \phi^{(i)}(t) \phi^{(i)}(t - n) dt = \delta[n]$$

**then**

$$\int_{-\infty}^{\infty} \phi^{(i+1)}(t) \phi^{(i+1)}(t - n) dt = \delta[n]$$

**Our initial guess for the iteration was  $\phi^{(0)}(t) = \text{box on } [0,1]$  which is orthonormal w.r.t. its shifts. By induction, we have  $\int_{-\infty}^{\infty} \phi(t) \phi(t - n) dt = \delta[n]$  as the limit, provided that the cascade algorithm converges.**

12

**Note: with the alternating flip requirement, which was necessary for the highpass filter in the case of orthogonal filters, we can show that**

$$\int_{-\infty}^{\infty} w(t) w(t - n) dt = \delta[n]$$

and

$$\int_{-\infty}^{\infty} \phi(t) w(t - n) dt = 0$$

13

### Orthogonality in the Frequency Domain

Let

$$a[n] = \int_{-\infty}^{\infty} \phi(t) \phi(t - n) dt$$

Use Parseval's theorem

$$a[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\Omega) \overline{\hat{\phi}(\Omega)} e^{-i\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\phi}(\Omega)|^2 e^{i\Omega n} d\Omega$$

Trick: split integral over entire  $\Omega$  axis into a sum of integrals over  $2\pi$  intervals

$$a[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} |\hat{\phi}(\Omega + 2\pi k)|^2 e^{i(\Omega + 2\pi k)n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2 e^{i\Omega n} d\pi$$

14

Take the Discrete Time Fourier Transform of both sides

$$A(\omega) \equiv \sum_n a[n]e^{-i\omega n} = \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

If  $\phi(t - n)$  are orthonormal, then  $a[n] = \delta[n]$

$$\Rightarrow A(\omega) = 1$$

So the scaling function and its shifts are orthogonal if

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = 1$$

Note: if we set  $\omega = 0$ , then

$$\sum_k |\hat{\phi}(2\pi k)|^2 = 1$$

and since  $\hat{\phi}(0) = 1$ , we see that

$$\hat{\phi}(2\pi k) = 0 \text{ for } k \neq 0$$

15

Connection between orthogonal wavelets and orthogonal filters (in frequency domain):

Start with an orthogonal scaling function:

$$1 = \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

and then change scale using the refinement equation in the frequency domain:

$$\hat{\phi}(\Omega) = H_0(\Omega/2) \hat{\phi}(\Omega/2)$$

$$1 = \sum_{k=-\infty}^{\infty} |H_0(\Omega/2 + \pi k)|^2 |\hat{\phi}(\Omega/2 + \pi k)|^2$$

16



**Trick: split sum into even and odd**

$$\begin{aligned} 1 &= \sum_{k=-\infty}^{\infty} |H_0(\Omega/2 + 2\pi k)|^2 |\hat{\phi}(\Omega/2 + 2\pi k)|^2 + \\ &\quad \sum_{k=-\infty}^{\infty} |H_0(\Omega/2 + \pi(2k + 1))|^2 |\hat{\phi}(\Omega/2 + \pi(2k + 1))|^2 \\ &= |H_0(\Omega/2)|^2 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega/2 + 2\pi k)|^2 + \\ &\quad |H_0(\Omega/2 + \pi)|^2 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega/2 + \pi + 2\pi k)|^2 \end{aligned}$$

**But each of the two infinite sums is equal to 1**

**So the discrete time filter  $H_0(\omega)$  must be orthogonal:**

$$1 = |H_0(\omega)|^2 + |H_0(\omega + \pi)|^2$$