

Appendix C

Stationary phase

See Stein's book *Harmonic analysis* [?], chapter 8, as a reference on stationary phase and for proofs of the claims below.

If an integrand has a phase factor with no stationary points, and the amplitude is otherwise smooth, then the integral has a very small value because the positive parts cancel out the negative parts. The following result makes this heuristic precise as an asymptotic bound on the value of the integral when the phase has a large prefactor.

Lemma 4. (*The non-stationary phase lemma.*) Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\phi \in C^\infty(\overline{\text{supp } \chi})$, and let

$$I_\alpha = \int_{\mathbb{R}^n} e^{i\alpha\phi(x)} \chi(x) dx.$$

If $\nabla\phi(x) \neq 0$ for all $x \in \text{supp } \chi$, then

$$|I_\alpha| \leq C_m \alpha^{-m}, \quad \text{for all } m > 0.$$

Proof. Integrate by parts after inserting an m -th power of the differential operator

$$L = \frac{I - \Delta_x}{1 + \alpha^2 |\nabla_x \phi(x)|^2},$$

which leaves the exponential factor unchanged. A fortiori, $1 + \alpha^2 |\nabla_x \phi(x)|^2 > C\alpha^2$ for some number $C > 0$. Deal with the odd values of m by interpolation (geometric mean) from the $m - 1$ and $m + 1$ cases. \square

If the phase otherwise has critical points, then the value of the integral is mostly determined by the behavior of the integrand near those critical points.

Lemma 5. *Consider the same setting as earlier, but consider the presence of a point x^* such that*

$$\nabla\phi(x^*) = 0, \quad D^2\phi(x^*) \text{ invertible,}$$

where $D^2\phi$ denotes the Hessian matrix of ϕ . Assume that $\nabla\phi(x) \neq 0$ for $x \neq x^*$. Then, as $\alpha \rightarrow \infty$,

$$I_\alpha = \left(\frac{2\pi}{\alpha}\right)^{n/2} \chi(x^*) e^{i\alpha\phi(x^*)} \frac{e^{i\frac{\pi}{4} \text{sgn}(D^2\phi(x^*))}}{\sqrt{\det(D^2\phi(x^*))}} + O(\alpha^{-\frac{n}{2}-1}),$$

where sgn denotes the signature of a matrix (the number of positive eigenvalues minus the number of negative eigenvalues.)

See [?] for a proof. More generally, if there exists a point x^* where all the partials of ϕ of order less than or equal to ℓ vanish, but $\partial^\ell\phi(x^*)/\partial x_1^\ell \neq 0$ in some direction x_1 , then it is possible to show that $I_\alpha = O(\alpha^{-1/\ell})$.

Here are a few examples.

- A good example for the above lemma is

$$\int_{-\infty}^{\infty} e^{i\alpha x^2} dx \sim \frac{1}{\sqrt{\alpha}}.$$

The real part of the integrand, $\cos(\alpha x^2)$, is non-oscillatory at the origin, but develops significant oscillations as soon as x is on the order of $\pm 1/\sqrt{\alpha}$. The extent of the range over which the integrand essentially does not oscillate (e.g., as measured from the length of the first half period) determines the order of magnitude of the value of the integral.

- An important case not immediately handled by any of the previous lemmas is the stationary phase explanation of the often-invoked fact that¹

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i(y-x)\cdot\xi} d\xi \right) f(x) dx \sim f(y).$$

The large factor α of the stationary phase lemmas can be placed in the exponent as $i\alpha(y-x)\cdot\xi$. The rescaling $\xi' = \alpha\xi$ quickly helps to get

¹The actual value of the integral is $(2\pi)^n f(y)$. The function f is only required to be continuous with some decay at infinity for this relation to make sense pointwise. Fourier analysis makes all of this precise, of course.

rid of it by turning it into a multiplicative $1/\alpha^n$ factor for the integral above. Hence the equivalent, stationary-phase-friendly formulation of the relation above is really

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i\alpha(y-x)\cdot\xi} d\xi \right) f(x) dx \sim \frac{f(y)}{\alpha^n}.$$

As a function of ξ alone, the phase $\phi(\xi) = (y-x)\cdot\xi$ has a critical point when $x = y$, but the Hessian is degenerate: $\phi''(\xi) = 0$. We cannot apply any of the stationary phase lemmas to the integral on ξ alone.

The solution is to consider the double integral over x and ξ : the phase $\phi(x, \xi) = (y-x)\cdot\xi$ is still critical when $x = y$, and now $\xi = 0$, but the Hessian matrix is

$$D^2\phi = \begin{pmatrix} \nabla_x \nabla_x \phi & \nabla_x \nabla_\xi \phi \\ \nabla_\xi \nabla_x \phi & \nabla_\xi \nabla_\xi \phi \end{pmatrix} = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

which is invertible independently of the base point (x^*, ξ^*) . Hence lemma 5 applies in $2n$ dimensions, and actually predicts the exact value of the integral, namely $(2\pi/\alpha)^n f(y)$. The condition $y = x$ signifies that, of all the values of $f(x)$, only that at $x = y$ matters for the result of the integral. The condition $\xi = 0$ is a manifestation of the fact that $f(x)$ was assumed to be minimally smooth (hence it is $\hat{f}(0)$ when $\xi = 0$ that matters). The function f may have oscillatory factors like $e^{i100\psi(x)}$ for some other phase ψ , but no factors of the form $e^{i\alpha\psi(x)}$ involving α explicitly.

- Another interesting example is the integral

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i(y-x)\cdot\xi} d\xi \right) \int e^{ix\cdot\eta} F(\eta) d\eta dx$$

which often appears in Fourier analysis. It can be seen as the composition of an inverse Fourier transform of F , from η to x , followed by a Fourier transform, from x to ξ , followed by an inverse Fourier transform, from ξ to y . Indeed, the integral reduces to (an unimportant multiple of 2π times) $\check{F}(y)$. For fixed η we can still see the phase as having two arguments, namely $\phi(x, \xi) = (y-x)\cdot\xi + x\cdot\eta$, but the equations for the critical points now look more symmetric:

$$\frac{\partial\phi}{\partial x} = \eta - x = 0, \quad \frac{\partial\phi}{\partial \xi} = y - x = 0,$$

and $D^2\phi$ is the same as previously. We now have $x^* = y$ and $\xi^* = \eta$, so $\phi(x^*, \xi^*) = y \cdot \eta$. Stationary phase over the inner (x, ξ) variables then reduces the outer η integral to (a constant times) $\int e^{iy \cdot \eta} F(\eta) d\eta$, as needed.

The relation $\eta = \xi$ indicates that, in the course of the first two Fourier transforms taking η to x , then to ξ , it is only the value of F at $\eta = \xi$ which matters to determine the result $F(\xi)$. The relation $x = y$ indicates that, from the result $f(x)$ of having done the first Fourier transform from η to x , it is only the value $f(y)$ at $x = y$ which matters to determine the end result $f(y) = \check{F}(y)$.

The set of equations

$$x = y, \quad \xi = \eta$$

is a simple example of a so-called canonical relation in phase-space, the space made of all the quadruples $(x, \xi; y, \eta)$. In particular, it is precisely the relation corresponding to the identity map from (x, ξ) to (y, η) . The adjective “canonical” refers to the fact that the map is symplectic, i.e., preserves areas, which is instantiated in our context by the fact that $|\det D^2\phi| = 1$. Phase-space relations are introduced and used in chapter 8.

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