Pak
10/19/05

Cor p-prime $n=4p-1$ $K=2p-1$
$\mathcal{F}=\{A_1,\dots A_m\}$, $A_i\subset[n]$, $|A_i|=K$, $|A_i\cap A_j|\neq p-1$
$\Rightarrow$ $m\leq 2\binom{n}{p-1}<1.8^n$

$\chi(\mathbb{R}^n,\delta):=$ min# colors to color $\mathbb{R}^n$ s.t. no two points at distance δ are colored the same
(so $\chi(\mathbb{R}^n,\delta)=\chi(\mathbb{R}^n,1)$ $\forall\,\delta>0$)

Thm (FW) : $\chi(\mathbb{R}^n,1)>(1+\varepsilon)^n$ for some $\varepsilon>0$, n suff large

pf : Note $\chi(\mathbb{R}^n,\delta)>\chi(H)$ where
$V(H)\subset\mathbb{R}^n$, $E(H)=\{uv\mid u,v\in V(H),\ d(u,v)=\delta\}$
Then let, for $n=4p-1$ p a prime, $S_p=\{0,1\}^n\cap\{|x|=2p-1\}$
Given $A\subset[n]$, let $\tilde{A}\in\mathbb{R}^n$ be incidence vector
$\text{dist}(\tilde{A},\tilde{B})^2=|A|+|B|-|A\cap B|$. Let $\delta=\sqrt{3p-1}$
(so $\text{dist}(\tilde{A},\tilde{B})=\delta\Leftrightarrow|A\cap B|=p-1$. Look at
$H_p=$ graph of δ-distances in S_p. Cor $\Rightarrow$
$\alpha(H_p)\leq 2\binom{n}{p-1}$, so $\chi(H_p)\geq |H|/\alpha(H_p)\geq \dfrac{\binom{n}{2p-1}}{2\binom{n}{p-1}}\geq(1+\varepsilon)^n$

This is enough for all n, since $\exists$ inf. many primes fairly close together + $\chi(\mathbb{R}^n,1)\geq\chi(\mathbb{R}^{n'},1)$ for some ε
✓

Borsuk Conj.
$X\subset\mathbb{R}^n$ convex set, $\text{diam}(X)=1$. Then $\exists\, X_i$ s.t.
$X=\coprod_{i=1}^{n+1}X_i$ s.t. $\text{diam}(X_i)<1$

Thm : true for $n=2$ (Borsuk), $n=3$ (somebody), ~~not for~~

Thm ~~[illegible]~~ (Kalai + Kahn)
min # parts $>(1+\varepsilon)^{\sqrt{n}}$ for n large enough

Pf: They found a new realization of H_p in $\mathbb{R}^{n^2}$
(the trick was doing it so that δ is now the largest distance)

$A \subset [n]$, $|A| = k$ $\quad n = 4p-1 \quad k = 2p-1$

$A \to \hat{A} \in \{0,1\}^d \quad d = \binom{n}{2}$

$\hat{A} = (a_{\{i,j\}}) \quad a_{\{i,j\}} = \begin{cases} 1 & \text{if } i \in A, j \notin A \\ 0 & o/w \end{cases}$

So $\hat{A}$ has $k(n-k)$ ones, $|A \cap B| = r$

$\text{dist}(\hat{A}, \hat{B})^2 = \cancel{2k(n-k) - |A\cap B||A^c \cap B^c|}$

$2k(n-k) - r(n-2k+r) + (k-r)^2$

So to make dist max set r close to $k - \frac{n}{4}$, i.e. $p-1$, Hooray!

Now have large # points at maximum distance, so can't split into subsets w/ less distance