Pak
10/7/05

Crapo's (hah!) Bijection

$n(F)=|F|-|V|+c(F)$

$$\sum_{t\in G} (x+1)^{\alpha(t)} (y+1)^{\beta(t)} = \sum_{F\subseteq E} x^{c(F)-c(E)} y^{n(F)}$$

Idea of natural bijection (details totally unimportant)

$(x+1)^{\alpha(t)} = \sum_{i=0}^{\alpha(t)} \binom{\alpha(t)}{i} x^i$, $(y+1)^{\beta(t)} = \sum_{i=0}^{\beta(t)} \binom{\beta(t)}{i} y^i$

So take subset of internally active edges + remove them, subset of external + add to get bijection between $\varphi: (\underset{\text{tree}}{t}, \underset{\text{inact. set}}{A}, \underset{\text{ex. act. set}}{B}) \longrightarrow t-A+B$

This is a bijection, proof that it is is left to us

φ preserves statistics: $\alpha(t)-|A| = c(F)-c(E)$, $F=\varphi(t,A,B)$ $|B|=n(F)$

Clumsy: $\sum_{t\subseteq G} x^{\alpha_<(t)} y^{\beta_<(t)} = \sum_{t'\subseteq G} x^{\alpha_{<'}(t')} y^{\beta_{<'}(t')}$

$<$, $<'$ two diff't orders. Make φ: trees $\to$ trees

$\varphi(t)=t'$, $\alpha_<(t)=\alpha_{<'}(t')$, $\beta_<(t)=\beta_{<'}(t')$

don't try this on HW

So find B's trick on page 353 change order for two edges $\Rightarrow$ ✓ + can change easily w/out affecting count much, it's all easy...

Def'n medial graph $H=H(G)$, G planar connected graph, build H by adding vertices one on each edge, add edges between those that are adjacent in circular order $\Rightarrow$ 4-regular graph (possibly w/ loops). Now cut each vertex in two along orig edge or $\perp$ to it $\Rightarrow$ 2-regular $\Rightarrow$ $\coprod$ cycles

w.c. = white cycles
b.c. = black cycles
C = cut, i.e. way of cutting vertices

Say a cycle is "white" if it has an original vertex inside of it, black o/w

Look at poly $\underset{F_G(x,y)}{} \sum_{\text{cut } C} x^{\#\text{white}(C)-1} y^{\#\text{black}(C)} = T_G(x+1, y+1)$

Pf: Look at bijection between cuts + subsets of edges $\Psi(C) = \{e \in G \mid e \text{ cut } \perp\}$ then check that everything works

Reidemeister moves