Pak
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Thm (Schur) (~1915)
$\forall r\ \exists n=n(r)$ s.t. in every r-coloring of $[n]$
$\exists$ a monoch'c Schur triple $\{x, y, x+y\}$
e.g. $r=1$ $n=3$
$r=2$ $n=9$ or something
In fact, $n(r) \le e \cdot r!$

Proof (Jukna's book) (p.326)
$\chi : [n] \to [r]$. Pick the most popular color c_1
Let $A_1 = \{x_0 < \dots < x_{m_1}\} = \chi^{-1}(c_1)$ $m_1+1 \ge \frac{n}{r}$
$B_1 = \{x_1 - x_0, \dots x_{m_1} - x_0\}$ must have no c_1 (if $x_i = 2x_0$, leave out)
$A_2 \subseteq B_1$, c_2 most popular color in B_1, $A_2 = \chi^{-1}(c_2) \cap B_1$
$= \{y_0 \dots y_{m_2}\}$, etc. repeat r times. Eventually
you eliminate all colors ~~but one~~ ✓

Thm (van der Woerden) (~1928)
$\forall r, l\ \exists n=n(r)$ s.t. $\forall$ r-coloring of $[n]$ $\exists$
monochromatic arithmetic progression of length l
I.e. Let $H_{n,l}$ be hypergraph on n vertices
w/ edges being length l arith. sequences, then
$\chi(H_{n,l}) \xrightarrow[n\to\infty]{} \infty$

Pf: Induction on ℓ. ~~$\ell = 2$~~ $\ell = 2$ $n(K,2) = K+1$

for $\ell \Rightarrow$ for $\ell+1$. Now. Take n large enough, break $[n]$ into $\geq 2m_1$ intervals of length t_1. $\exists\ r^{t_1}$ colorings of each interval. Let $m_1 = n(r^{t_1}, \ell)$. So $\exists$ arith seq of same-coloring intervals of length ℓ.

$n \geq 2m_1 t_1$. Now let $t_1 \geq 2m_2 t_2$, ~~divide~~ ~~into~~ divide each interval of that sequence into $2m_2$ intervals of length t_2, $m_2 = n(r^{t_2}, \ell)$ so $\exists$ identically colored progression, etc.

Repeat r times, w/ $t_r \geq 1$. OKay,

Now look at $a_{11\ldots1}, a_{11\ldots1\,\ell+1}, a_{11\ldots1\,\ell+1\,\ell+1}$, etc.,

i.e. numbers in first slot of all list of intervals in first of all but last (in which it's the end), first of all but last 2, where it's the end, etc.

This is $K+1$ numbers, so two are colored the same, say $a_{\underbrace{1\ldots1}_{p}\underbrace{\ell+1\ldots\ell+1}_{r-p}}$ and $a_{\underbrace{\ldots}_{q}\underbrace{\ldots}_{r-q}}$

Then progression is

$$\boxed{a_{\underbrace{1\ldots1}_{p}\underbrace{i\ldots i}_{q-p}\underbrace{\ell+1\ldots\ell+1}_{r-q}} \quad 1 \leq i \leq \ell+1}$$