

Review/recap of theory so far.

Evolution of wave profile, as given by the characteristic solution.

Graphical interpretation:

- Move each point on graph at velocity  $c(\rho)$ . Evolution as sliding of horizontal slabs at different velocities (guarantees conservation).
- Causes wave steepening, wave breakdown, and multiple values.
- Mathematical model breakdown. Quasi-equilibrium assumption fails and PDE model breaks down.
- Back to "physics". Need to augment model with new physics, namely: shocks in the case of traffic flow or flood waves.

First look at EXPANSION FANS: A discontinuity in the initial, or boundary, conditions gives rise to an expansion fan if the edge characteristics do not cross and leave a gap. That is:  $c$  is increasing across the discontinuity:

DEFINITION: An expansion fan is the solution produced by a collection of characteristics, all starting at one single point, but with a range of values for the solution there.

First look at SHOCKS: A discontinuity in the initial, or boundary, conditions gives rise to shock if the edge characteristics cross. That is:  $c$  is increasing across the discontinuity:

EXAMPLE: green-light turns red problem. One shock on each side of light.

Shock on the right: last car through light.

Shock on the left: locus where cars break behind the light.

Shock conditions:

- Rankine-Hugoniot shock speed  $S = [q]/[\rho]$ .
- Derive from conservation:
- In shock frame: flow from the left  $= \rho_-(u_- - S) = F_-$
- In shock frame: flow from the right  $= \rho_+(u_+ - S) = F_+$
- Then conservation is  $F_- = F_+$ , which gives  $S = [q]/[\rho]$ .

Graphical interpretation of Rankine-Hugoniot: slope of secant line in  $\rho$ - $q$  diagram.

Note: limit for infinitesimal strength shocks is characteristic speed.

EXAMPLES with EXPANSION FANS

1.  $u_t + u^*u_x = 0$ , with  $u(x, 0) = 0$  for  $x < 0$ ,  
a) and  $u(x, 0) = 1$  for  $x > 0$ .
2.  $u_t + u^*u_x = -u$ , with  $u(x, 0) = 0$  for  $x < 0$ ,  
a) and  $u(x, 0) = 1$  for  $x > 0$ .
3.  $u_t + u^*u_x = 0$ , with  $u(x, 0) = 1$  for  $x > 0$ ,  
a)  $u(0, t) = 1$  for  $0 < t < 1$ ,

4.  $u_t + u^*u_x = 0$ , with  $u(x, 0) = 0$  for  $x > 0$ ,
- a)  $u(0, t) = 1$  for  $0 < t < 1$ ,
  - b) and  $u(0, t) = 0$  for  $1 < t$ .

IMPORTANT point: When dealing with shocks, it is CRUCIAL TO KNOW WHAT IS CONSERVED.

Note that the equation  $u_t + u^*u_x = 0$  can arise from various conservation laws:

- Conserved density  $\rho = u$ , flux  $q = (1/2)*u^2$ ,
- Conserved density  $\rho = u^2$ , flux  $q = (2/3)*u^3$
- Conserved density  $\rho = u^3$ , flux  $q = (3/4)*u^4$

Each of this gives rise to the same solution as long as there are no shocks, but give rise to DIFFERENT shock speeds, hence different solutions once shocks arise.

Example:  $u_t + u^*u_x$ , with IC:  $u = 1$   $x < 1$ , and  $u = 0$  for  $x > 0$ .  
Compute solution for the three cases above.

RECAP: IMPORTANT point concerning the ENTROPY condition:

We need shocks to stop the crossing of the characteristics: characteristics end at shock, and do not continue #A on the other side. Hence crossings are avoided. This requires the condition  $c_- > S > c_+$  "Entropy condition".

Why name? Note that #A implies that INFORMATION IS LOST AT SHOCKS, and problem becomes IRREVERSIBLE once SHOCKS FORM. Hence #A is determining the arrow of time. In physics, information content is measured by the entropy, that must be non-decreasing (second law of thermodynamics). Hence the name.

Example: look at red light turns green problem.

In addition to the solution with the rarefaction fan, in principle the problem admits a solution with a "shock", but this shock does not satisfy entropy, and in fact generates information (characteristics). This solution is NOT "physical" and should be considered "in-admissible" in the augmented model.

Back to entropy condition and R-H jump condition:

$$c_- > S > c_+ \quad \text{and} \quad S = [q]/[\rho].$$

Inspect graphical meaning of this in  $\rho$ - $q$  diagram.

Consistent for traffic flow and river flows ( $q$  concave and convex).

Leads to:

- For traffic flow  $\rho$  increases across shock.
- For river flow  $A$  decreases across shock.

IMPORTANT NOTE ABOUT THE ENTROPY CONDITION:

It applies if one assumes standard drivers and standard driving conditions. Using "special" drivers one can arrange to have discontinuities that do not satisfy the entropy condition.

1. Example (traffic flow): Situation at the start of a car race, with all the racing cars neatly organized in a pack behind a lead car. This gives rise a "square" wave density shape. The entropy condition is violated by the front discontinuity. But this is "car ballet", not traffic flow.
2. Example (traffic flow): driver that goes slower than the road conditions allow, creating long line of cars behind [common in mountain roads]. Again, it requires a special driver to maintain the discontinuity at the front of the pack. But the discontinuity at the end of the pack is a standard shock.
3. Example (river flow): push water from behind with a paddle. This is the equivalent of example 2 above in traffic flow.

### SOLUTION PROCESS FOR KINEMATIC CONSERVATION LAWS THAT SUPPORT SHOCKS

- (A) Solve for the characteristics starting at EVERY point along the curves with data (initial, boundary, whatever). If the data is given with different formulas for various segments, solve each segment separately.
- (B) Check for GAPS in the characteristics, caused by jumps (discontinuity in the data), and fill-in the corresponding fans of characteristics.
- (C) Inspect the set of characteristics thus obtained and check for crossings. Use shocks to eliminate the crossings (with the techniques to be described below). Shocks used ONLY to prevent characteristics from crossing. Characteristics converge into them and STOP there!
- (D) After inserting the shocks, and solving for their paths in space-time, check again characteristics. Make sure that all crossings have been resolved, and that (for each shock) you do have the characteristics on each side ending there. Shocks MUST satisfy both the Rankine-Hugoniot jump conditions and the entropy condition.
- (E) Note that a shock may pass through different "regions" where the solution on each side is given by changing formulas (characteristics starting at different parts of data). Keep this in mind when solving for their paths.
- (F) Check for possible shock "collisions" and resolve them.

### INFORMATION LOSS AT SHOCKS.

For more details see the two problems:

- Information loss and traffic flow shocks.
- Preventive driving and dissipation.

in the problem series: KiNe (Kinematic Waves)

Example: Can one calculate how much "information" is lost at a shock?

Answer depends on how one measures the amount of information that one has.

However, a "partial" answer to this question goes as follows:

1) As long as the solution has derivatives,  $\rho_t + q(\rho)_x = 0$  implies the conservation of any  $f(\rho)$ , with flux  $F(\rho)$  given by the equation  $F' = c \cdot f'$ .

2) When shocks appear, this is no longer true. Hence one can measure the "loss" caused by a shock by picking an "appropriate"  $f(\rho)$ . Notice that, if one knows  $\int f(\rho) dx$  for all possible  $f$ 's, one can recover  $\rho$  (provided  $\rho$  is, say, continuous).

3) Example:  $A_t + ((1/2) \cdot A^2)_x = 0$ , with  $A$  conserved [baby example of a river floods equation]. Show then:

$$\frac{d}{dt} \left( \int_a^b (1/2) \cdot A^2 dx \right) = (1/3) \cdot A^3(a, t) - (1/3) \cdot A^3(b, t) - (1/12) [A]^3$$

$$\underbrace{\hspace{10em}}_{\text{Conservative part (a flux)}} \quad \underbrace{\hspace{10em}}_{\text{Stuff lost at shock.}}$$

where  $[A] = A_- - A_+ > 0$  is the jump across the shock in  $A$ .

Note that  $\int A^2$  is a rough measure of the amount of information, in the following sense: for fixed total area  $\int A$  in some interval, and  $A > 0$ , the minimum value for  $\int A^2$  occurs when  $A$  is constant. The more structure (info) the function  $A$  has, the larger  $\int A^2$  will be.

Motivate definition of "example entropy" as  $\int u^2$

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