18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Exam Number 02

18.306 — MIT (Fall 2009)

Rodolfo R. Rosales

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Due: Last day of lectures.

Contents

1.1 Statement: Green's functions (problem 01).

The Green's function for the heat equation in the infinite line solves the problem

$$G_t = G_{xx}, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \tag{1.1}$$

with the initial condition
$$G(x,\, 0) = \delta(x), \tag{1.2}$$
where $\delta(\cdot)$ denotes Dirac's delta function. In addition, G is bounded[1] for any $t > 0$.

For any positive constant $\nu > 0$, $\boldsymbol{\nu\, \delta(\nu\, x) = \delta(x)}$. It follows that: If G is a solution, then $\nu\, G(\nu\, x,\, \nu^2\, t)$ is a solution.

[1] This guarantees uniqueness. The heat equation has some very badly behaved solutions if some restriction such as boundedness is not imposed.

Hence, from uniqueness, for any $\nu > 0$,
$$G(x,\, t) \equiv \nu\, G(\nu\, x,\, \nu^2\, t). \tag{1.3}$$
Thus
$$G(x,\, t) = \frac{1}{\sqrt{t}}\, g\left(\frac{x}{\sqrt{t}}\right), \tag{1.4}$$
for some function $g = g(\xi)$ — Proof: substitute $\nu = 1/\sqrt{t}$ in (1.3).

Notice that, from the equation — assuming that G_x vanishes as $x \to \pm\infty$ — it follows that $\int_{-\infty}^{\infty} G(x,\, t)\, dx$ is a constant, independent of time. Hence, from the initial conditions
$$1 = \int_{-\infty}^{\infty} G(x,\, t)\, dx. \tag{1.5}$$

Tasks:

1. Substitute (1.4) into (1.1), and get an o.d.e. for g.

2. Solve the o.d.e. for g, and thus find the Green's function.

Hints: (i) *g is even, since (1.1 – 1.2) is invariant under reflection across the t-axis.* **(ii)** *Since G vanishes as $t \downarrow 0$ for any fixed $x \neq 0$, $g(\xi)$ vanishes, as $\xi \to \pm\infty$, faster than $1/\xi$.* **(iii)** *(1.5) must apply.* **(iv)** *The o.d.e. for g is second order. It can be integrated once, to yield a first order equation.* **(v)** *The general solution for the o.d.e. that g satisfies has two constants. These can be determined using either (i) and (iii), or (ii) and (iii) — (i) and (ii) turn out to be equivalent.*

1.2 Statement: Green's functions (problem 03).

Find the Green's function for the initial value problem for the heat equation with mixed (as stated below) boundary conditions in an interval. Namely, solve the problem
$$\boldsymbol{G_t = G_{xx}} \quad \text{for} \quad \boldsymbol{0 < x < 1} \quad \text{and} \quad \boldsymbol{t > 0,} \quad \text{with} \tag{1.6}$$

(a) Boundary conditions $\boldsymbol{G_x = 0}$ at $\boldsymbol{x = 0}$ and $\boldsymbol{G = 0}$ at $\boldsymbol{x = 1}$.
(b) Initial conditions $\boldsymbol{G(x,\, 0) = \delta(x - y)}$, where $\boldsymbol{0 < y < 1}$, and $\boldsymbol{\delta} =$ Dirac's delta function.

This problem can be done by the method of images, using the Green's function for the infinite line
$$G_\infty(x,\, y,\, t) = \frac{1}{2\sqrt{\pi\, t}}\, \exp\left(-\frac{(x-y)^2}{4\, t}\right). \tag{1.7}$$

Note that:

1. **Let u solve the heat equation, with $u_x = 0$ at $x = 0$, for all times. Then u is even.**
   Proof: From the equation, $u_{xt} = u_{xxx}$. Thus $u_{xxx} = 0$ at $x = 0$ for all t. Again, from the equation, $(u_{3x})_t = u_{5x}$. Hence $u_{5x} = 0$ at $x = 0$ for all t. Continue the argument, and show that all the odd derivatives of u vanish. Hence u is even. The proof of item **2** is identical.
2. **Let u solve the heat equation, with $u = 0$ at $x = 0$, for all times. Then u is odd.**
3. **If u is a solution of the heat equation with even initial data, then u is even.**
4. **If u is a solution of the heat equation with odd initial data, then u is odd.**
5. **If u is a solution of the heat equation with periodic initial data, then u is periodic.**
   Proof of 3–5: use uniqueness, and the symmetries of the equation and data.

HINT: *Use* **1-5** *to replace the problem in (1.6) by one on the infinite line, periodic (which period?) and with the appropriate odd/even properties. Then use (1.7) to solve this new problem.*

1.3 Statement: Green's functions (problem 04).

Find the Green's function for the initial value problem for the heat equation with Robin (as stated below) boundary conditions in the semi-infinite line. Namely, solve the problem

$$\boldsymbol{G_t = G_{xx}} \quad \text{for} \quad \boldsymbol{x > 0} \quad \text{and} \quad \boldsymbol{t > 0,} \quad \text{with} \tag{1.8}$$

(a) Boundary condition $\boldsymbol{G - G_x = 0}$ at $\boldsymbol{x = 0}$.
(b) Initial conditions $\boldsymbol{G(x, 0) = \delta(x - y)}$, where $\boldsymbol{0 < y}$ and $\boldsymbol{\delta}$ = Dirac's delta function.
(c) $\boldsymbol{G}$ is bounded for any $\boldsymbol{t > 0}$.

This problem can be done by the method of images, using the Green's function for the infinite line

$$\left.\right\} \quad \begin{aligned} G_\infty(x, y, t) &= G_\infty(x - y, t) \\ &= \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right). \end{aligned} \tag{1.9}$$

Note that, if u solves the heat equation, with $u - u_x = 0$ at $x = 0$, then:

1. $\boldsymbol{v = u - u_x}$ **solves the heat equation with $v = 0$ at $x = 0$.**

2. $\displaystyle \boldsymbol{u = e^x \int_x^\infty e^{-s}\, v(s, t)\, ds.}$

HINT: (I) *Use* **1-2** *to replace the problem for* G *by one solvable using the method of images — recall that,* $v_t = v_{xx}$ *and* $v = 0$ *at* $x = 0$ *means that* v *is odd.* (II) *Use (1.9) to solve this new problem.*

Remark 1.1 *You have to be* careful with Robin boundary conditions. *For example:*

$$T_t = T_{xx} \quad \textit{for} \quad x > 0 \;\; \textit{and} \;\; t > 0, \quad \textit{with}$$

$T + T_x = 0$ *at* $x = 0$*, has the solutions* $\boldsymbol{T = e^{-x+t}}$*, which are well behaved in space, but grow exponentially in time. The reason is that the condition* $T_x = -T$ *leads to a run-away heating: the hotter it gets at the origin, the larger the heat flow across there is. Physically, this is non-sense.*

1.4 Statement: Green's functions (problem 05).

The Green's function for the heat equation half plane Dirichlet[2] signaling problem is defined by

$$G_t = G_{xx}, \quad \text{for} \quad -\infty < t < \infty \quad \text{and} \quad x > 0, \tag{1.10}$$

with the boundary condition $$G(0,\, t) = \delta(t), \tag{1.11}$$
where $\delta(\cdot)$ denotes Dirac's delta function. In addition: G is bounded away from the origin $(0,\, 0)$ and satisfies **causality** . . . . **$G = 0$ for $t < 0$.** Thus **we only need to find** G **for $t \geq 0$ only.**

For any constant $\nu \neq 0$, $\boldsymbol{\nu^2\, \delta(\nu^2\, t) = \delta(t)}$. Thus: If G is a solution, $\nu^2\, G(\nu\, x,\, \nu^2\, t)$ is a solution.

Hence, from uniqueness, for any $\nu \neq 0$, $$G(x,\, t) = \nu^2\, G(\nu\, x,\, \nu^2\, t). \tag{1.12}$$

Set $\nu = x/t$ to get, for some function $g = g(\xi)$, $$G(x,\, t) = \frac{1}{t}\, g\left(\frac{x^2}{t}\right). \tag{1.13}$$

Notice that

A. $\boldsymbol{g(0) = 0}$. This follows because, for any $t > 0$ fixed, G must vanish as $x \downarrow 0$.

B. $\displaystyle \boldsymbol{\int_0^\infty g(\xi)\, \frac{d\xi}{\xi} = 1.}$ We have $\displaystyle \int_{-\infty}^\infty G(x,\, t)\, dt = \int_0^\infty g(\xi)\, \frac{d\xi}{\xi} = \text{constant}$, for any $x > 0$.

By taking $x \downarrow 0$, and using (1.11), the result follows.

– **Task 1:** Substitute (1.13) into (1.10), and get an o.d.e. for g.

[2] Temperature prescribed on the boundary.

– **Task 2:** Solve the o.d.e. for g, and thus find the Green's function.

Hints: (i) *The o.d.e. for g is second order. It can be integrated once, to yield a first order equation.* **(ii)** *The general solution to the o.d.e. in* **1** *has two constants. These follow from* **A – B** *above.*

Remark 1.2 *For any fixed $x > 0$, the formula in (1.13) should, as $t \downarrow 0$, smoothly match with the solution $G \equiv 0$ for $t < 0$. This will be guaranteed by the fact that $g(\xi)$ vanishes exponentially as $\xi \to \infty$. Hence, G as given by (1.13), as well as all its derivatives, vanish as $t \downarrow 0$ for any $x > 0$.*

THE END.