18.305 Exam 2,

December 6, 04. Closed Book.

1. Find, for $\lambda >> 1$, the leading term for each of the following integrals:

a. $\int_1^2 e^{-\lambda(x^4+x^2)}dx$. (15 points.)

b. $\int_{-\infty}^{\infty} e^{i\lambda(x^4+x^2)}dx$. (15 points.)

c. $\int_{-\infty}^{\infty} e^{i\lambda \sinh x}dx$. (bonus 20 points.)

2. Consider the differential equation

$\epsilon y'' - 3(\tan x)y' + y = 0.$

Solve this equation with the following boundary conditions:

a. $y(1/2) = 2, y(1) = 1$. Find the approximate solution for $1/2 < x < 1$.(15 points.)

b. $y(0) = 0, y(1) = 1$. Find the approximate solution for $0 < x < 1$. (15 points.)

c. $y(-1) = 1, y(1) = 0$.Find the approximate solution for $-1 < x < 1$. (10 points.)

Indicate the location of boundary layer or layers and give the order of magnitude of the width for each of them.

3. Find, with the two-scale method, the lowest-order approximate solution of

$\ddot{y} + (1 + \epsilon \dot{y}^2)y = 0,$

satisfying

$y(0) = 0,\ \dot{y}(0) = 1.$

For what values of t is this approximate solution good? (30 points)

Formulae which may be useful:

$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$,

$\int_{-\infty}^{\infty} e^{ix^2}dx = \sqrt{\pi}\, e^{i\pi/4}$,

$\cosh z = (e^z + e^{-z})/2,\ \sinh z = (e^z - e^{-z})/2,$

$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y.$

Also, the solution for

$\epsilon y'' + \alpha x y' + \beta y = 0$

is

$$y = e^{-\alpha x^2/(4\epsilon)}\left[aD_\nu\left(\sqrt{\tfrac{|\alpha|}{\epsilon}}\, x\right) + bD_\nu\left(-\sqrt{\tfrac{|\alpha|}{\epsilon}}\, x\right)\right],$$

where

$$\nu = \frac{\beta}{|\ \alpha\ |} - \frac{sign(\alpha) + 1}{2}.$$

Solutions

1a. Let
$v(x) = x^4 + x^2.$
The minimum of $v(x)$ inside the interval $[1, 2]$ is at the lower endpoint $x_0 = 1$. We have
$v(1) = 2,\ v'(1) = 6.$
Thus, near $x = 1$,
$v(x) \simeq 2 + 6(x - 1).$
Therefore,
$I(\lambda) \simeq e^{-2\lambda} \int_1^2 e^{-6\lambda(x-1)} dx \simeq \dfrac{e^{-2\lambda}}{6\lambda}.$

1b. Let
$u(x) = x^4 + x^2.$
The origin is the only point of stationary phase. Near the origin,
$u(x) \simeq x^2.$
Thus we have
$I(\lambda) \simeq \int_{-\infty}^{\infty} e^{i\lambda x^2} dx = \sqrt{\dfrac{\pi}{\lambda}}\, e^{i\pi/4}.$

1c. Since the integrand has no finite endpoints nor points of stationary phase, the dominant contributions to this integral come from point or points in the complex plane. Since the integrand is of the order of unity, we expect that the integral vanishes exponentially as $\lambda \to \infty$.

As the integrand has no singularities, the dominant contributions come from one or more saddle points in the complex plane. Let
$f(z) = \sinh z.$
The saddle points are the zeroes of
$f'(z) = \cosh z.$
Thus a saddle point z_0 satisfies
$e^{z_0} + e^{-z_0} = 0,$
which can be writeen as
$e^{2z_0} = -1.$
Therefore,
$z_0 = \pm i\pi/2, \pm 3i\pi/2, \cdots \pm(2n+1)\pi/2 \cdots.$
There are an infinite number of saddle points.

To see which one or ones contribute, we evaluate the integrand at each of the saddle points. We have
$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y.$
Thus
$\sinh(\pm i\pi \dfrac{2n+1}{2}) = \pm i(-1)^n.$
The integrand is exponentially large at
$z_0 = -i\pi/2, 3i\pi/2, -5i\pi/2, 7i\pi/2 \cdots,$

which are eliminated as points of contribution.

To eliminate the other saddle points, we look to deform the contour of integration originally on the entire real axis. The goal is to make the deformed contour pass through one or more of the saddle points, with the magnitude of the integrand largest at the saddle points.

In order to jusify the deformation of the contour of integration, we must make sure that the integrand vanishes at the added contour at infinity. Since

$| \ e^{i\lambda \sinh(x+iy)} \ | = e^{-\lambda \cosh x \sin y}$,

the integrand vanishes as $x \to \pm\infty$ provided that $\sin y > 0$. Therefore, we may move the contour up to the horizontal line $y = \pi/2$, (but not to the horizontal line $y = -3\pi/2$, for example.)

On the line $y = \pi/2$,

$\sinh z = \sinh(x + i\pi/2) = i \cosh x.$

Hence we have

$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda \cosh x} dx.$

The function $\cosh x$ is smallest at $x_0 = 0$ (note that this is the saddle point $z_0 = i\pi/2$). Thus the integrand is sharply peaked around this point. Applying the Laplace method yields

$I(\lambda) \simeq \int_{-\infty}^{\infty} e^{-\lambda(1+x^2/2)} dx = e^{-\lambda} \sqrt{\frac{2\pi}{\lambda}}$.

2a. Since $a(x) = -3 \tan x < 0$, $1/2 < x < 1$, the rapidly varying solution is increasing. There is a boundary layer of width ϵ near the upper endpoint $x = 1$.

We have

$y_{out}(x) = e^{\int \frac{dx'}{3 \tan x'}} = 2(\sin x / \sin \frac{1}{2})^{1/3}$,

which satisfies the boundary condition $y(1/2) = 2$. The solution $y_{out}(x)$ is a good approximation everywhere outside of the boundary layer near $x = 1$.

Near the upper endpoint $x = 1$, we have

$\epsilon y_{in}'' - 3(\tan 1) y_{in}' = 0.$

Thus

$y_{in}(x) = [1 - y_{out}(1)] e^{-(3 \tan 1) \frac{1-x}{\epsilon}} + y_{out}(1).$

From $y_{out}(x)$ given above, we get

$y_{out}(1) = 2^{4/3} (\cos \frac{1}{2})^{1/3}.$

2b. Since $a(x) = -3 \tan x \leq 0$, $0 < x < 1$, the rapidly varying solution is increasing with x. There is a boundary layer of width ϵ near the upper endpoint $x = 1$.

Since $a(0) = 0$, there is a boundary layer of width $\sqrt{\epsilon}$ near $x = 0$.

Near $x = 0$, we have

$y^{(a)}(x) = e^{3x^2/(4\epsilon)} \Big[A D_{1/3}(\sqrt{\frac{3}{\epsilon}}\, x) + B D_{1/3}(-\sqrt{\frac{3}{\epsilon}}\, x) \Big].$

The rapidly varying solution is practically zero at $x = 0$, being appreciable only near $x = 1$. This rapidly varying solution joins with the second term for $y^{(a)}(x)$ given above. Thus we have

$B = 0.$

The boundary condition

$y(0) = 0 \Rightarrow A = 0.$

Thus the solution inside the boundary layer near $x = 0$ is zero.

Matching $y_{out}(x)$ with this solution, we find that

$y_{out}(x) = 0.$

Near $x = 1,$ we have

$y(x) = e^{-\frac{3\tan 1(1-x)}{\epsilon}},$

which satisfies the boundary condition $y(1) = 1$. This solution is exponentially small outside of the boundary layer near $x = 1$. Hence it approximate the solution well everywhere in the interval $0 < x < 1$.

2c. As we know, the solution of an interior boundary layer with a negative α is generally a U-shape curve, being very small everywhere except in regions of widths ϵ near the endpoints.

In the present problem, the boundary condition at the upper endpoint is $y(1) = 0$. Thus the solution is small everywhere except near the lower endpoint. We get

$y = e^{-\frac{3\tan 1(1+x)}{\epsilon}}$

which satisfies the boundary condition at $x = -1$ exactly, and the boundary condition at $x = 1$ up to an exponentially small amount.

3. Let

$y(t) = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots,$

where

$\tau = \epsilon t.$

We have,

$$(\frac{\partial^2}{\partial t^2} + 1 + 2\epsilon\frac{\partial^2}{\partial t\partial\tau} + \epsilon^2\frac{\partial^2}{\partial\tau^2})(y_0 + \epsilon y_1 + \cdots)$$
$$+\epsilon(\frac{\partial}{\partial t}y_0 + \epsilon\frac{\partial}{\partial\tau}y_0 + \epsilon\frac{\partial}{\partial t}y_1 + \cdots)^2(y_0 + \epsilon y_1 + \cdots) = 0.$$

The initial conditions are

$y_0(0,0) = y_1(0,0) = \cdots = 0, \quad \frac{\partial}{\partial t}y_0 \mid_{t=\tau=0} = 1, \quad \frac{\partial}{\partial t}y_1 \mid_{t=\tau=0} = -\frac{\partial}{\partial\tau}y_0 \mid_{t=\tau=0}, etc.$

It is straightforward to find that

$y_0(t,\tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}.$

The initial condition $y_0(0,0) = 0$ give

$A(0) = -A^*(0).$

And the initial condition $\frac{\partial}{\partial t}y_0 \mid_{t=\tau=0} = 1$ gives

$iA(0) - iA^*(0) = 1.$

Thus we get

$A(0) = -i/2.$

The equation satisfied by y_1 is

$(\frac{\partial^2}{\partial t^2} + 1)y_1 + 2\frac{\partial^2}{\partial t\partial\tau}y_0 + (\frac{\partial}{\partial t}y_0)^2 y_0 = 0.$

Setting the sum of secular terms in this equation to zero, we get

$2iA' + A^2A^* = 0.$

Let

$A = e^{i\theta}R.$

We have

$2i(R' + i\theta' R) + R^3 = 0.$

Setting the imaginary part of the equation above to zero, we get

$R(\tau) = 1/2,$

a constant, where the initial condition has been taken into account. Setting the real part of the equaton above to zero, we get

$\theta' = 1/8.$

Thus

$\theta = -\pi/2 + \tau/8.$

We get

$A(\tau) = -i\frac{e^{i\tau/8}}{2}.$

Thus

$y_0(t) = \sin(1 + \frac{\epsilon}{8})t.$

This approximate solution is good for $t = O(\epsilon^{-1})$.

Comments: The two-scale method yields an approximate solution good up to $t = O(\epsilon^{-1})$. We shall show that the method of renormalized perturbation yields an approximate solution good for all t.

Let the angular frequency of the system be ω, which we express as

$\omega^2(1 + a_1\epsilon + a_2\epsilon^2 + \cdots) = 1.$

The coefficients $a_1, a_2 \cdots$ will be chosen to eliminate the secular terms. The equation can be written as

$\ddot{y} + \omega^2 y = -\epsilon\, \dot{y}^2\, y - \omega^2(a_1\epsilon + a_2\epsilon^2 + \cdots)y.$

The last term in the differential equation above is the counter term. We put

$y(t) = y_0(t) + \epsilon y_1(t) + \cdots$

and the differential equation for y becomes

$(\frac{d^2}{dt^2} + \omega^2)(y_0 + \epsilon y_1 + \cdots) = -\epsilon(\dot{y}_0 + \epsilon\, \dot{y}_1 + \cdots)^2(y_0 + \epsilon y_1 + \cdots)$
$-\omega^2(a_1\epsilon + a_2\epsilon^2 + \cdots)(y_0 + \epsilon y_1 + \cdots).$

The initial conditions are

$y_0(0) = y_1(0) = \cdots = 0, \quad \dot{y}_0\,(0) = 1, \quad \dot{y}_1\,(0) = 0,$ etc.

It is straightforward to find that

$y_0(t) = i\frac{e^{i\omega t} - e^{-i\omega t}}{2\omega}.$

Note that the coefficient of $e^{i\omega t}$ and that of $e^{-i\omega t}$ in the expression above are both purely imaginary.

The equation for $y_1(t)$ is

$(\frac{d^2}{dt^2} + \omega^2)y_1 = -\,\dot{y}_0^2\, y_0 - \omega^2 a_1 y_0.$

We shall show that all secular terms of this equation for y_1 can be eliminated by a choice of a_1.

We have

$\dot{y}_0\,(t) = -\frac{e^{i\omega t} + e^{-i\omega t}}{2}.$

Note that the coefficient of $e^{i\omega t}$ and that of $e^{-i\omega t}$ of $\dot{y}_0\,(t)$ are real. Thus $\dot{y}_0^2$ is a linear superposition of $e^{im\omega t}$ terms with real coefficients. Similarly, all $e^{im\omega t}$ coefficients of $\dot{y}_0^2\, y_0$ are imaginary.

The differential equation for y_1 is

$(\frac{d^2}{dt^2} + \omega^2)y_1 = i\frac{e^{3i\omega t} - e^{-3i\omega t} + e^{i\omega t} - e^{-i\omega t}}{8\omega} + \omega a_1 i\frac{e^{i\omega t} - e^{-i\omega t}}{2}.$

Setting the sum of the coefficients of $e^{i\omega t}$ in the equation above to zero. we get

$a_1 = -\frac{1}{4\omega^2}.$

We see that the sum of the $e^{-i\omega t}$ coefficients also vanishes with this choice of $a_{1.}$

That a single choice of a_1 eliminates two kinds of secular terms is due to the fact that $a_1$is real.

With a_1 real, the right-side of the differential equation above is real. This ensures that the sum of the coefficients of $e^{-i\omega t}$ terms is equal to the complex conjugate of that of the $e^{i\omega t}$ terms. Consequently, the sum of the $e^{-i\omega t}$ coefficients vanishes if that of the $e^{i\omega t}$ coefficients does.

With this value of a_1, we get

$$\omega = 1 + \frac{\epsilon}{8} + \cdots,$$

and

$$y_0(t) = \frac{\sin(1 + \frac{\epsilon}{8})t}{1 + \frac{\epsilon}{8}},$$

which, in the lowest-order of ϵ and with t as large as ϵ^{-1}, agrees with the result obtained earlier with the two-scale method.

The $e^{im\omega t}$ coefficients of $y_1(t)$ are imaginary. To see this, we solve the differential equation satisfied by $y_1(t)$ and get

$$y_1(t) = Be^{i\omega t} + B^* e^{-i\omega t} + \frac{e^{3i\omega t} - e^{-3i\omega t}}{64\omega^3 i}.$$

The initial condition of $y_1(0) = 0$ leads to B being purely imaginary. From the initial condition $\dot{y}_1(0) = 0,$ we find that the value of B is $3i/(64\omega^3)$. Thus

$$y_1(t) = i\frac{3e^{i\omega t} - 3e^{-i\omega t} - e^{3i\omega t} + e^{-3i\omega t}}{64\omega^3}.$$

We note that all coefficients of the $e^{im\omega t}$ terms in the expression above are imaginary. This, in turn, enables us to show that the secular terms in the differential equation for $y_2(t)$ can be eliminated by a choice of a_2.

Instead of showing this, we shall, instead, show by induction that the secular terms in the differential equation for y_{n+1} can be eliminated by the counter terms, where $n = 2, 3 \cdots$.

Let $y_l(t),\ l = 0, 1 \cdots n,$ be a superposition of $e^{im\omega t}$ terms with imaginary coefficients. Then the $e^{im\omega t}$ coefficients of $\dot{y}_l(t),\ l = 0, 1 \cdots n$ are real. With the same arguments as presented before, we may show that it is possible to eliminate both the $e^{i\omega t}$ secular terms and the $e^{-i\omega t}$ secular terms in the differential equation for $y_{n+1}(t)$ with a single choice of a_n. Also, since $y_{n+1}(0) = 0$, the $e^{im\omega t}$ coefficients of $y_{n+1}(t)$ are imaginary. Thus we have proven by induction that it is possible to eliminate all secular terms with the counter terms.