18.305 Fall 2004/05

Assignment 8 Solutions: The Two-scale Method

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1. **Solve with the two-scale method**

$$\frac{d^2y}{dt^2} + \epsilon(1+y^2)\frac{dy}{dt} + y = 0, \ \epsilon << 1 \tag{1}$$

with $y(0) = 0$, $\dot{y}(0) = 1$.

For what values of t do you expect the approximate solution to be good. Can you explain why the solution you obtained satisfies

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y = 0 \tag{2}$$

as $t \to \infty$.

2. **Apply the two-scale method to the problem**

$$\ddot{x} + x = \epsilon(\dot{x} - \frac{1}{3}\dot{x}^3) \tag{3}$$

with $x(0) = 1$ and $x(0) = a$.

Can you explain why the solution always approaches a limit cycle as $t \to \infty$?

Solutions:

1. A regular perturbation analysis gives that it is convenient to use $\tau = \epsilon t$ as a second scale in the treatment of the problem. We then remember

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial \tau} \\ \frac{d^2}{dt^2} &= \frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial^2}{\partial t\partial \tau} + \epsilon^2\frac{\partial^2}{\partial \tau^2} \end{aligned} \tag{4}$$

Pluggin' in these identities into (1), along with

$$y = y_0 + \epsilon y_1 + ... \tag{5}$$

where the quantities are considered functions of both the variables t and τ, we obtain

$$[\frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial^2}{\partial t\partial \tau} + \epsilon^2\frac{\partial^2}{\partial \tau^2} + 1](y_0 + \epsilon y_1 + ...) = -\epsilon[1 + (y_0 + \epsilon y_1 + ...)^2](\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial \tau})(y_0 + \epsilon y_1 + ...)$$

The initial conditions translate into

$$\begin{aligned} (y_0 + \epsilon y_1 + ...)|_{(0,0)} &= 0, \ i = 0, 1, 2, .. \\ (\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial \tau})(y_0 + \epsilon y_1 + ...)|_{(0,0)} &= 1 \end{aligned}$$

which gives

$$y_n(0,0) = 0, \text{ for all } n = 0,1,2,..$$

$$\frac{\partial}{\partial t} y_0(0,0) = 1,$$

$$\left(\frac{\partial y_n}{\partial \tau} + \frac{\partial y_{n+1}}{\partial t}\right)|_{(0,0)} = 0, \text{for all } n = 0,1,2,..$$

We now look at the order 1 terms to see

$$\frac{\partial^2}{\partial t^2} y_0 + y_0 = 0$$

which implies

$$y_0 = B(\tau)e^{it} + C(\tau)e^{-it}$$

where B and C are arbitrary functions. Making use of the initial conditions, we see that we can write

$$y_0 = A(\tau)e^{it} + A^*(\tau)e^{-it} \tag{6}$$

with $A(0) = \frac{1}{2i}$. We next examine the order ϵ terms-

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} + 1\right)y_1 &= -2\frac{\partial^2}{\partial t \partial \tau} y_0 - (1 + y_0^2)\frac{\partial}{\partial t} y_0 \\ &= -2iA'e^{it} + 2iA^{*\prime}e^{-it} - [1 + (Ae^{it} + A^*e^{-it})^2](iAe^{it} - iA^*e^{-it}) \\ &= -2iA'e^{it} + 2iA^{*\prime}e^{-it} - i[1 + (A^2e^{2it} + 2AA^* + A^{*2}e^{-2it})](Ae^{it} - A^*e^{-it}) \end{aligned}$$

The secular terms on the right hand side of this last equality are seen to be

$$-i(2A^{'} + A + A^2A^*)e^{it} + i(2A^{*'} + A^* + AA^{*2})e^{-it}$$

We observe that the second summand is just the complex conjugate of the first, hence to eliminate all the secular terms it suffices to choose A such that

$$2A^{'} + A + A^2A^* = 0$$

To solve this, we let $A = Re^{i\theta}$, which leads to

$$2(R' + i\theta' R) + R + R^3 = 0$$

Equating the real and imaginary parts to zero,

$$\begin{aligned} \theta' &= 0 \\ 2R' + R + R^3 &= 0 \end{aligned}$$

Since $A(0) = 1/2i$, we have $\theta(0) = -\pi/2$, which implies $\theta(\tau) = -\pi/2$ for all τ. To solve the differential equation for R, we let

$$U = R^2$$

(this is not necessary, but it makes the algebra simpler), then

$$U' = 2RR' = -(R^2 + R^4) = -(U + U^2)$$

$$\frac{U'}{U(U+1)} = \frac{U'}{U} - \frac{U'}{U+1} = -1$$

which gives, by integration,

$$\frac{U}{U+1} = ce^{-\tau}$$

Making use of the initial condition $U(0) = R^2(0) = 1/4$, we find $U = \frac{1}{5e^{\tau}-1}$, hence

$$A = \frac{1}{\sqrt{5e^{\tau}-1}}$$

Thus, from (6)

$$y \approx y_0 = \frac{2\sin t}{\sqrt{5e^{\epsilon t}-1}}$$

The solution obtained by the two-scale method is a good approximation for times of $O(1/\epsilon)$, as we may have secular terms of order $\epsilon^n t^{n-1}$ from the contribution of y_n to the series (5). However, for this particular example, further analysis(similar to the one on pp328 of the textbook) shows that the obtained solution is a good approximation for all times.

We observe that $y \to 0$ as $\tau \to \infty$, hence the y^2 term in the differential equation becomes much smaller than the other terms. That is why the solution y_0 satisfies the differential equation (2) as $\tau \to \infty$.

2. A regular perturbation analysis of the problem shows the existence of secular terms in the form ϵe^{it}, therefore it is appropriate to use $\tau = \epsilon t$ as a second time scale for this problem. Using the identities (4), and

$$x = x_0 + \epsilon x_1 + ...$$

we obtain

$$[\frac{\partial^2}{\partial t^2}+2\epsilon\frac{\partial^2}{\partial t\partial\tau}+\epsilon^2\frac{\partial^2}{\partial\tau^2}+1](x_0+\epsilon x_1+...) = \epsilon[(\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial\tau})(x_0+\epsilon x_1+...)-\frac{1}{3}((\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial\tau})(x_0+\epsilon x_1+...))^3] \tag{7}$$

The initial conditions translate into

$$\begin{aligned} (x_0 + \epsilon x_1 + ...)|_{(0,0)} &= 0, \ i = 0, 1, 2, .. \\ (\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial\tau})(x_0 + \epsilon x_1 + ...)|_{(0,0)} &= a \end{aligned}$$

which gives

$$x_n(0,0) = 0, \text{ for all } n = 0, 1, 2, ..$$

$$\frac{\partial}{\partial t}x_0(0,0) = 1,$$

$$(\frac{\partial x_n}{\partial\tau} + \frac{\partial x_{n+1}}{\partial t})|_{(0,0)} = 0, \text{for all } n = 0, 1, 2, ..$$

We now look at the order 1 terms to see

$$\frac{\partial^2}{\partial t^2}x_0 + x_0 = 0$$

which implies

$$x_0 = B(\tau)e^{it} + C(\tau)e^{-it}$$

where B and C are arbitrary functions. Making use of the fact that x_0 can be chosen to be a real solution, we see that we can write

$$x_0 = A(\tau)e^{it} + A^*(\tau)e^{-it} \tag{8}$$

with $A(0) = \frac{a}{2i}$. We next examine the $O(\epsilon)$ in the differential equation (7),

$$\begin{aligned}
(\frac{\partial^2}{\partial t^2} + 1)x_1 &= -2\frac{\partial^2}{\partial t \partial \tau}x_0 + [\frac{\partial}{\partial t}x_0 - \frac{1}{3}(\frac{\partial}{\partial t}x_0)^3] \\
&= -2iA'e^{it} + 2iA^{*\prime}e^{-it} + [(iAe^{it} - iA^*e^{-it}) - \frac{1}{3}(iAe^{it} - iA^*e^{-it})^3] \\
&= -2iA'e^{it} + 2iA^{*\prime}e^{-it} + [iAe^{it} - iA^*e^{-it} + \frac{1}{3}i(A^3e^{3it} - 3A^2A^*e^{it} + 3AA^{*2}e^{-it} - A^{*3}e^{-}
\end{aligned}$$

The secular terms on the right hand side are seen to be

$$i(-2A' + A - A^2A^*)e^{it} - i(-2A^{*\prime} + A^* - A^{*2}A)e^{-it}$$

We again see that the second summand is just the complex conjugate of the first, so to eliminate all the secular terms we only need to choose A such that

$$-2A' + A - A^2A^* = 0$$

To solve this, we let $A = Re^{i\theta}$, which leads to

$$-2(R' + i\theta' R) + R - R^3 = 0$$

Equating the real and imaginary parts to zero,

$$\begin{aligned}
\theta' &= 0 \\
-2R' + R - R^3 &= 0
\end{aligned}$$

Since $A(0) = 1/2$, we have $\theta(0) = -\pi/2$, which implies $\theta(\tau) = -\pi/2$ for all τ. To solve the differential equation for R, we let

$$U = R^2$$

(this is not necessary, but it makes the algebra simpler), then

$$U' = 2RR' = R^2 - R^4 = U - U^2 = -U(U-1)$$

$$-\frac{U'}{U(U-1)} = -\frac{U'}{U} + \frac{U'}{U-1} = 1$$

which gives, by integration,

$$\frac{U-1}{U} = ce^{-\tau}$$

Making use of the initial condition $U(0) = R^2(0) = a^2/4$, we find

$$U = \frac{a^2}{(1 - e^{-\tau})a^2 + 4e^{-\tau}}$$

and $A = \frac{a}{\sqrt{(1-e^{-\tau})a^2+4e^{-\tau}}}$, hence

$$x \approx x_0 = A(\tau)\frac{e^{it}}{i} - A^*(\tau)\frac{e^{-it}}{i} = \frac{2a \sin t}{\sqrt{(1 - e^{-\epsilon t})a^2 + 4e^{-\epsilon t}}} = \frac{2a \sin t}{\sqrt{a^2 + (4 - a^2)e^{-\epsilon t}}}$$

which is a good approximation to the actual solution for at least the times of order $\frac{1}{\epsilon}$. As is easily seen, the solution approaches to the limiting function $y = 2 \sin t$, no matter what the value of the parameter a is, unless a is exactly zero.

From a physical point of view, this can be explained as follows. We rewrite the differential equation in the form

$$\ddot{x} + x = \epsilon\dot{x}(1 - \frac{1}{3}\dot{x}^2) \tag{9}$$

This models an oscillator with damping factor $\epsilon(1 - \frac{1}{3}\dot{x}^2)$. Damping is positive or negative depending on the value of $|\dot{x}|$. A small solution will be damped positively, since $1 - \frac{1}{3}\dot{x}^2$ will be positive. Similarly, a large solution will be damped negatively. In this case,a limiting solution, which attracts all the initial conditions, is natural to expect.