18.305 Fall 2004/05

Solutions to Assignment 5: The Stationary Phase Method

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1. **Find the leading term for each of the integrals below for $\lambda >> 1$.**

   (a) $\int_{-1}^{1} e^{i\lambda t^3} dt$

   (b) $\int_{1}^{\infty} e^{i\lambda t^2} dt$

   (c) $\int_{0}^{\pi} e^{i\lambda \cos t} dt$

2. **Find the leading term for each of the integrals below $\lambda >> 1$.**

$$I(x) = \int_{-\infty}^{\infty} e^{ixt} e^{it^5/5} dt \quad (1)$$

   Consider both cases in which $x > 0$ or $x < 0$.

   Solutions:

1. **(a)** $I(\lambda) = \int_{-1}^{1} e^{i\lambda t^3} dt$

Can write

$$I(\lambda) = \int_0^1 e^{i\lambda t^3} dt + \int_{-1}^{0} e^{i\lambda t^3} dt$$

We observe that the second integral is the complex conjugate of the first, hence

$$I(\lambda) = 2\,\mathrm{Re} \int_0^1 e^{i\lambda t^3} dt$$

Moreover, we see that the point $t = 0$ is the only stationary phase point, which gives the main contribution to the integral, therefore we can make the approximation

$$I(\lambda) \approx 2\,\mathrm{Re} \int_0^\infty e^{i\lambda t^3} dt = 2\,\mathrm{Re}\, J(\lambda)$$

where

$$J(\lambda) = \int_0^\infty e^{i\lambda t^3} dt$$

Note that we do not need to calculate the end point contributions, which would come out to be of smalller order than the contribution of the stationary phase point-we are only asked the leading term.

The integrand of J is oscillatory. We would like to change the contour into one on which is exponentially decreasing, for then we we would be able to express it as a gamma function. Thus we put

$$z^3 = it^3$$

where t is real. This means the path over which the integral is taken would be

$$z = i^{1/3} t$$

There are three roots for $i^{1/3}$: $e^{i\pi/6}$, $e^{5i\pi/6}$, $e^{3i\pi/2}$. An analysis of those paths brings that we can only change the domain of integral, which is originally the real axis, into the straightline making an angle $\pi/6$, which is the only case where we can close our contour with a zero contribution arc. Thus

$$J = e^{i\pi/6}\int_0^{\infty} e^{-\lambda t^3}dt$$

Making one more variable change

$$\tau = \lambda t^3$$

we arrive at

$$J = \frac{e^{i\pi/6}}{3\lambda^{1/3}}\int_0^{\infty} e^{-\tau}\tau^{-2/3}d\tau = \frac{e^{i\pi/6}}{3\lambda^{1/3}}\Gamma(1/3)$$

Therefore, the answer is given by

$$I(\lambda) \approx \frac{1}{3\lambda^{1/3}}\Gamma(1/3)$$

1. **(b)** $\int_1^{\infty} e^{i\lambda x^2}dx$

There is no stationary phase points in the domain of the integral, hence this is an integral in the form (8.36) of the book, i.e of the form

$$I(\lambda) = \int_a^b e^{i\lambda u(x)}h(x)dx \qquad (2)$$

for which the end point contributions are important. Hence the leading form can be given by (8.39) in the book, which is

$$\frac{e^{i\lambda u(b)}h(b)}{i\lambda u'(b)} - \frac{e^{i\lambda u(a)}h(a)}{i\lambda u'(a)}$$

In our case, we have only one end point, hence the leading term is

$$-\frac{e^{i\lambda u(1)}h(1)}{i\lambda u'(1)} = -\frac{e^{i\lambda}}{2i\lambda}$$

1. **(c)** $\int_0^{\pi} e^{i\lambda \cos x}dx$

The stationary phase points are solutions to $\sin x = 0$, which are $x = 0, \pi$. Since stationary phase points contribute more than the end points, we do not consider end points for the purpose of calculating the leading term. In this example, the end points are stationary phase points, so they will already be taken care of when calculating the stationary phase points, so in some sense, we can say that there is no end point contribution.
We use the formula (8.45) and (8.46) from the book which are

$$e^{i\pi/4}\sqrt{\frac{2\pi}{\lambda u''(x_0)}}e^{i\lambda u(x_0)}h(x_0) \text{ and } e^{-i\pi/4}\sqrt{\frac{2\pi}{\lambda |u''(x_0)|}}e^{i\lambda u(x_0)}h(x_0) \qquad (3)$$

depending on the sign of $u''(x_0)$. Since the stationary phase points are end points, they contribute only half of the quantity they would if they were interior points. Thus we finally find

$$\frac{1}{2}e^{-i\pi/4}\sqrt{\frac{2\pi}{\lambda}}e^{i\lambda}+\frac{1}{2}e^{i\pi/4}\sqrt{\frac{2\pi}{\lambda}}e^{-i\lambda}=\sqrt{\frac{2\pi}{\lambda}}\cos(\lambda-\frac{\pi}{4})$$

as the leading term.

2. The integral (1) is in the form (2), with $u(t)=t$, $h(t)=e^{it^5/5}$. The integral has neither points of stationary phase nor finite end points. We first treat the simpler case $x<0$. We start with scaling the variable of integration by the transformation $t=(-x)^{1/4}z$, and the integral (1) becomes

$$I(x)=(-x)^{1/4}\int_{-\infty}^{\infty}e^{i\Lambda f(z)}dz$$

with $f(z)=-z+\frac{1}{5}z^5$ and $\Lambda=(-x)^{5/4}$. This last integrand has two stationary points $z=-1,1$, which are found by solving $f'(z)=-1+z^4=0$. Therefore the integral can be approximated by making use of (3), which give

$$e^{-i\pi/4}\sqrt{\frac{\pi}{2\Lambda}}e^{4i\Lambda/5}+e^{i\pi/4}\sqrt{\frac{\pi}{2}}e^{-4i\Lambda/5}$$

therefore

$$\begin{aligned}I(x) &= (-x)^{1/4}\sqrt{\frac{2\pi}{\Lambda}}\cos(\frac{4}{5}\Lambda-\frac{\pi}{4})\\ &= (-x)^{1/4}\sqrt{\frac{2\pi}{(-x)^{5/4}}}\cos(\frac{4}{5}(-x)^{5/4}-\frac{\pi}{4})=\sqrt{2\pi}(-x)^{-3/8}\cos(\frac{4}{5}(-x)^{5/4}-\frac{\pi}{4})\end{aligned}$$

For the case $x>0$, we scale the original integral with $t=x^{1/4}z$, and the integral (1) becomes

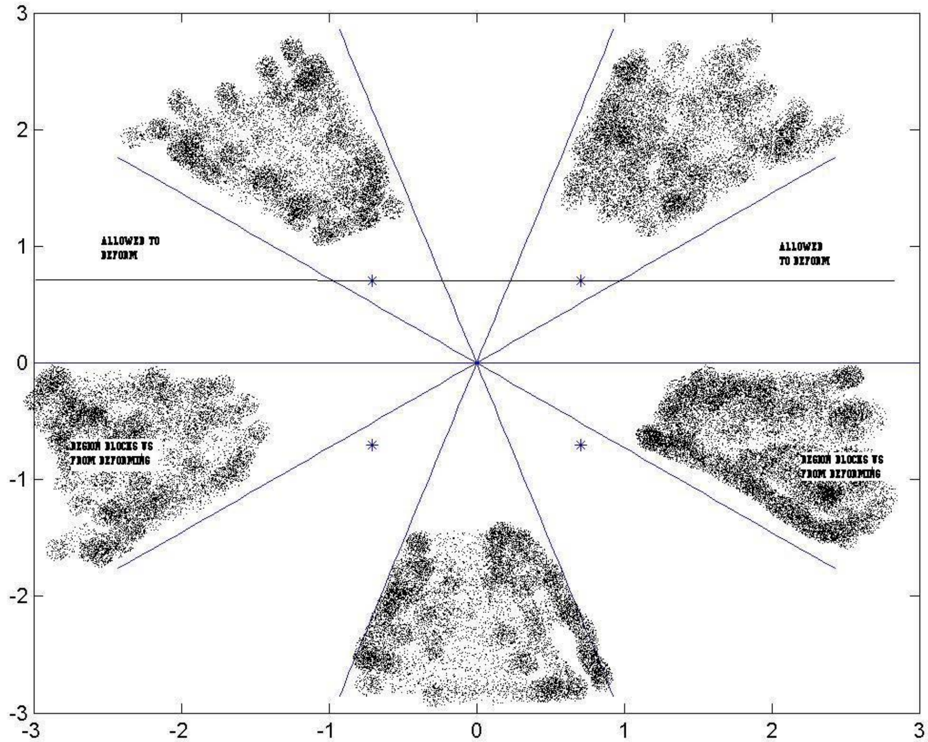
$$I(x)=x^{1/4}\int_{-\infty}^{\infty}e^{i\Lambda f(z)}dz \tag{4}$$

with $f(z)=z+\frac{1}{5}z^5$ and $\Lambda=x^{5/4}$. This integral still does not have any stationary phase points, hence we look at the critical points of f by solving $f'(z)=1+z^4=0$ which gives $z=e^{i\pi/4},e^{3i\pi/4},e^{5i\pi/4},e^{7i\pi/4}$.

Putting $z=re^{i\theta}$ in $f(z)=z+\frac{1}{5}z^5=r(\cos\theta+i\sin\theta)+\frac{1}{5}r^5(\cos 5\theta+i\sin 5\theta)$. So the integrand of (4) blows up for large r in the regions where

$$-\sin 5\theta>0$$

and becomes exponentially small in the regions where $\sin 5\theta>0$. These are shown in the figure below along with the critical points of f. The regions in which the integrand becomes exponentially large are shaded. We observe that we can deform the domain of our integral as shown in the figure, to upwards, so that the path of the integral now contains two of the critical points of f. We cannot deform the path of our integral


ALLOWED TO DEFORM
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REGION BLOCKS US FROM DEFORMING
3
2
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-3
-3
-2
-1
0
1
2
3

Figure 1

downwards, as the "black" regions, over which the integrand becomes exponentially large prevents us from doing so.

We calculate the relevant values

$$\begin{aligned} f(e^{i\pi/4}) &= \frac{4}{5}e^{i\pi/4}, \ f''(e^{i\pi/4}) = 4e^{3i\pi/4} \\ f(e^{3i\pi/4}) &= \frac{4}{5}e^{3i\pi/4}, \ f''(e^{3i\pi/4}) = 4e^{i\pi/4} \end{aligned}$$

which means we have the expansions

$$\begin{aligned} i\Lambda f(z) &\approx i\Lambda\frac{4}{5}e^{i\pi/4} - 2\Lambda e^{i\pi/4}(z - e^{i\pi/4})^2 \\ i\Lambda f(z) &\approx i\Lambda\frac{4}{5}e^{3i\pi/4} - 2\Lambda e^{-i\pi/4}(z - e^{3i\pi/4})^2 \end{aligned}$$

So the contributions from those critical points are calculated to be

$$x^{1/4}\sqrt{\frac{\pi}{2\Lambda e^{i\pi/4}}}\exp[i\Lambda\frac{4}{5}e^{i\pi/4}] + x^{1/4}\sqrt{\frac{\pi}{2\Lambda e^{-i\pi/4}}}\exp[i\Lambda\frac{4}{5}e^{3i\pi/4}]$$

which is

$$\sqrt{2\pi}x^{-3/8}\sqrt{2\pi}\exp(-\frac{2\sqrt{2}}{5}x^{5/4})\cos(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{1}{8}\pi)$$