Solutions to Problems for 2D & 3D Heat and Wave Equations

18.303 Linear Partial Differential Equations

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1 Problem 1

A rectangular metal plate with sides of lengths L, H and insulated faces is heated to a uniform temperature of u_0 degrees Celsius and allowed to cool with three of its edges maintained at 0^o C and the other insulated. You may use dimensional coordinates, with PDE

$$u_t = \kappa \nabla^2 u, \qquad 0 \leq x \leq L, \qquad 0 \leq y \leq H.$$

The BCs are

$$u(0, y, t) = 0 = u(L, y, t), \qquad u(x, 0, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, t) = 0. \tag{1}$$

(i) Solve for $u(x, y, t)$ subject to an initial condition $u(x, y, 0) = 100$.

(ii) Find the smallest eigenvalue λ and the first term approximation (i.e. the term with $e^{-\lambda\kappa t}$).

(iii) For fixed $t = t_0 \gg 0$, sketch the level curves $u =$ constant as solid lines and the heat flow lines as dotted lines, in the xy-plane.

(iv) Of all rectangular plates of equal area, which will cool the slowest? Hint: for each type of plate, the smallest eigenvalue gives the rate of cooling.

(v) Does a square plate, side length L, subject to the BCs (1) cool more or less rapidly than a rod of length L, with insulated sides, and with ends maintained at 0^o C? You may use the results we derived in class for the rod, without derivation.

Solution: (i) We separate variables as

$$u(x, y, t) = v(x, y)\, T(t)$$

to obtain

$$\frac{T'}{\kappa T} = \frac{\nabla^2 v}{v} = -\lambda \tag{2}$$

where λ is constant since the l.h.s. depends only on t and the middle depends only on (x, y). We so obtain the Sturm-Liouville problem

$$\nabla^2 v + \lambda v = 0 \tag{3}$$

The BCs are

$$v(0,y) = 0 = v(L,y), \qquad v(x,0) = 0, \qquad \frac{\partial v}{\partial y}(x,H) = 0. \tag{4}$$

Separating variables again for $v(x,y)$ gives

$$v(x,y) = X(x)\,Y(y) \tag{5}$$

and substituting into the PDE (3) gives

$$X''Y + XY'' + \lambda = 0$$

Rearranging gives

$$\frac{Y''}{Y} + \lambda = -\frac{X''}{X} = \mu \tag{6}$$

where μ is constant because the l.h.s. depends only on y and the middle on x. Introducing (5) into the BCs (4) gives

$$\begin{aligned}
0 &= v(0,y) = X(0)\,Y(y) \qquad &\Rightarrow \qquad X(0) = 0\\
0 &= v(L,y) = X(L)\,Y(y) \qquad &\Rightarrow \qquad X(L) = 0\\
0 &= v(x,0) = X(x)\,Y(0) \qquad &\Rightarrow \qquad Y(0) = 0\\
0 &= \frac{\partial v}{\partial y}(x,H) = X(x)\,Y'(H) \qquad &\Rightarrow \qquad Y'(H) = 0
\end{aligned}$$

We begin with the problem for $X(x)$, since we have a complete set (two) of homogeneous Type I BCs, the simplest to deal with. From (6), the ODE for $X(x)$ is

$$X'' + \mu X = 0; \qquad X(0) = 0 = X(L)$$

We've seen before that the eigenfunctions and eigenvalues are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \mu_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, 3...$$

From (6), with $\mu = \mu_n$, the ODE for $Y(y)$ is

$$Y'' + \nu Y = 0; \qquad Y(0) = 0 = Y'(H)$$

where $\nu = \lambda - \mu_n$. Again, we've seen ODEs like this before, that $\nu > 0$ and

$$Y(y) = A\cos\left(\sqrt{v}y\right) + B\sin\left(\sqrt{v}y\right)$$

Imposing the BCs gives

$$\begin{aligned} 0 &= Y(0) = A \\ 0 &= Y'(H) = B\sqrt{v}\cos\left(\sqrt{v}H\right) \end{aligned}$$

Note that B cannot be zero since we desire a non-trivial solution. Hence $\cos\left(\sqrt{v}H\right) = 0$ and

$$\sqrt{v}H = \left(m + \frac{1}{2}\right)\pi, \qquad m = 1, 2, 3...$$

Thus the eigenvalues and eigenfunctions are

$$\begin{aligned} v_m &= \left(m - \frac{1}{2}\right)^2 \frac{\pi^2}{H^2}, \qquad m = 1, 2, 3... \\ Y_m(y) &= \sin\left(\left(m - \frac{1}{2}\right)\pi\frac{y}{H}\right) \end{aligned}$$

The general solution to the Sturm-Liouville problem for $v(x, y)$ is

$$v_{mn}(x, y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\left(m - \frac{1}{2}\right)\pi\frac{y}{H}\right), \qquad n, m = 1, 2, 3...$$

From (2), the solution for $T(t)$ is

$$T = e^{-\kappa\lambda t} = e^{-\kappa(\nu_m + \mu_n)t} = \exp\left(-\left(\left(m - \frac{1}{2}\right)^2 \frac{1}{H^2} + \frac{n^2}{L^2}\right)\kappa\pi^2 t\right)$$

The general solution to the PDE and BCs for $u(x, y, t)$ is

$$u_{mn}(x, y, t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\left(m - \frac{1}{2}\right)\pi\frac{y}{H}\right)\exp\left(-\left(\left(m - \frac{1}{2}\right)^2 \frac{1}{H^2} + \frac{n^2}{L^2}\right)\kappa\pi^2 t\right),$$

for $n, m = 1, 2, 3...$ To satisfy the initial condition, we use superposition and sum over all m, n to obtain

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x, y, t) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\left(m - \frac{1}{2}\right)\pi\frac{y}{H}\right) \\ &\qquad \cdot\exp\left(-\left(\left(m - \frac{1}{2}\right)^2 \frac{1}{H^2} + \frac{n^2}{L^2}\right)\kappa\pi^2 t\right) \end{aligned}$$

To find the constants A_{mn}, we impose the IC:

$$100 = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\left(m-\frac{1}{2}\right)\pi\frac{y}{H}\right)$$

Using orthogonality relations for sin, we have

$$\begin{aligned}
A_{mn} &= \frac{4}{LH}\int_0^L\int_0^H 100\sin\left(\frac{n\pi x}{L}\right)\sin\left(\left(m-\frac{1}{2}\right)\pi\frac{y}{H}\right)dydx \\
&= \frac{400}{LH}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\int_0^H \sin\left(\left(m-\frac{1}{2}\right)\pi\frac{y}{H}\right)dy \\
&= \frac{400}{LH}\frac{L}{n\pi}\left(\cos(n\pi)-1\right)\frac{2H}{(2m-1)\pi}\left(\cos\left(\left(m-\frac{1}{2}\right)\pi\right)-1\right) \\
&= \frac{800}{n(2m-1)\pi^2}\left(1-(-1)^n\right)
\end{aligned}$$

Thus $A_{m2n} = 0$ and

$$A_{m2n-1} = \frac{1600}{(2n-1)(2m-1)\pi^2}$$

and

$$\begin{aligned}
u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1600}{(2n-1)(2m-1)\pi^2}\sin\left(\frac{(2n-1)\pi x}{L}\right)\sin\left(\frac{(2m-1)\pi y}{2H}\right) \\
&\qquad\cdot\exp\left(-\left(\frac{(2m-1)^2}{4H^2}+\frac{(2n-1)^2}{L^2}\right)\kappa\pi^2 t\right)
\end{aligned}$$

(ii) The smallest eigenvalue λ is

$$\lambda_{11} = \left(\frac{1}{4H^2}+\frac{1}{L^2}\right)\pi^2$$

and the first term approximation is

$$u(x,y,t) \approx u_{11}(x,y,t) = \frac{1600}{\pi^2}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{2H}\right)\exp\left(-\left(\frac{1}{4H^2}+\frac{1}{L^2}\right)\kappa\pi^2 t\right)$$

(iii) To draw the level curves, note that the isotherms (lines of constant temp, i.e. level curves) will intersect the insulated side ($y = H$) at right angles, and cannot intersect the other sides, which are also isotherms ($u = 0$). Using these constraints and continuity allow you to sketch the curves. The level curves and heat flow lines are plotted as solid and dotted lines, respectively, in Figure 1.

[extra] The heat flow lines are found by finding the orthogonal trajectories to the level curves

$$\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{2H}\right) = c$$

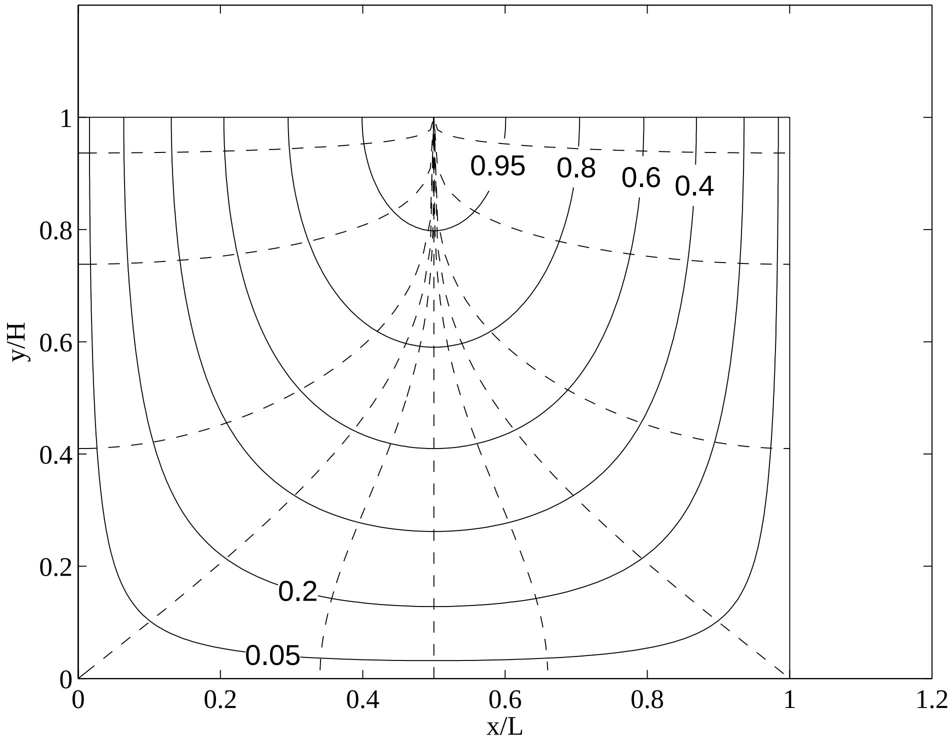

Figure 1: Sketch of level curves (solid) and heat flow lines (broken) of $u(x, y, t)$, scaled by its maximum.

Differentiating implicitly with respect to x gives

$$\cos\left(\frac{\pi x}{L}\right)\frac{\pi}{L}\sin\left(\frac{\pi y}{2H}\right)+\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{2H}\right)\frac{\pi}{2H}\frac{dy}{dx}=0$$

Solving for dy/dx yields

$$\frac{dy}{dx}=-\frac{2H}{L}\cot\left(\frac{\pi x}{L}\right)\tan\left(\frac{\pi y}{2H}\right)$$

The slope of the orthogonal trajectories are the negative reciprocal,

$$\frac{dy}{dx}=\frac{L}{2H}\tan\left(\frac{\pi x}{L}\right)\cot\left(\frac{\pi y}{2H}\right)$$

Thus

$$\tan\left(\frac{\pi y}{2H}\right)dy=\frac{L}{2H}\tan\left(\frac{\pi x}{L}\right)dx$$

Integrating gives

$$-\frac{2H}{\pi}\ln\left(\left|\cos\left(\frac{\pi y}{2H}\right)\right|\right)=-\frac{L}{\pi}\frac{L}{2H}\ln\left(\left|\cos\left(\frac{\pi x}{L}\right)\right|\right)+c$$

Thus

$$y=\frac{2H}{\pi}\cos^{-1}\left(A\left|\cos\left(\frac{\pi x}{L}\right)\right|^{L^2/4H^2}\right)$$

In Figure 1, we chose $H = L = 1$ and various values of A to obtain the dashed lines.

(iv) The rate of cooling is $\kappa\lambda_{11}$, and hence given a material (constant κ) of a certain fixed area $A = LW$, we can write λ_{11} as

$$\lambda_{11} = \left(\frac{L^2}{4A^2} + \frac{1}{L^2} \right) \pi^2$$

The rectangle of area A with the smallest rate of cooling is found by minimizing λ_{11} with respect to L:

$$\frac{d\lambda_{11}}{dL} = \left(\frac{2L}{4A^2} - \frac{2}{L^3} \right) \pi^2 = 0$$

so that

$$L^4 = 4A^2 \quad \Rightarrow \quad L = \sqrt{2A} \quad \Rightarrow \quad H = \frac{A}{L} = \sqrt{\frac{A}{2}}$$

Thus

$$\frac{H}{L} = \frac{\sqrt{A/2}}{\sqrt{2A}} = \frac{1}{2}$$

Thus the rectangle with these BCs (zero at all edges except the top) that cools the slowest is one with $H = L/2$, height half the width.

[optional] This makes sense, since when the sides are all held at zero the rectangle that cools the slowest is the square, and this can be constructed by joining two of the rectangles in this problem along their insulated sides.

(v) The smallest eigenvalue for the rod of length L with zero BCs at the ends is

$$\lambda_1 = \frac{\pi^2}{L^2}$$

and for the square of side length L subject to BCs (1) is

$$\lambda_{11} = \frac{5}{4}\frac{\pi^2}{L^2} > \lambda_1$$

Thus the rod cools more slowly, which makes sense since the square can cool from its lower horizontal side, unlike the rod which is insulated on its horizontal sides, and can only cool through its ends.

2 Problem 2

Haberman Problem 7.3.3, p. 287. Heat equation on a rectangle with different diffusivities in the x- and y-directions.

Solution: We solve the heat equation where the diffusivity is different in the x and y directions:

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$$

on a rectangle $\{0 < x < L, 0 < y < H\}$ subject to the BCs

$$\begin{aligned} u(0,y,t) &= 0, \qquad \frac{\partial u}{\partial y}(x,0,t) = 0, \\ u(L,y,t) &= 0, \qquad \frac{\partial u}{\partial y}(x,H,t) = 0, \end{aligned}$$

and the IC

$$u(x,y,0) = f(x,y).$$

We separate variables as before,

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

so that the PDE becomes

$$X(x)\,Y(y)\,T'(t) = k_1 X''(x)\,Y(y)\,T(t) + k_2 X(x)\,Y''(y)\,T(t)$$

Dividing by $X(x)\,Y(y)\,T(t)$ gives

$$\frac{T'(t)}{T(t)} = k_1 \frac{X''(x)}{X(x)} + k_2 \frac{Y''(y)}{Y(y)}$$

Since the l.h.s. depends on t and the r.h.s. on (x,y), both sides must equal a constant $-\lambda$ (we call this the separation constant):

$$\frac{T'(t)}{T(t)} = k_1 \frac{X''(x)}{X(x)} + k_2 \frac{Y''(y)}{Y(y)} = -\lambda$$

Thus

$$T(t) = ce^{-\lambda t}$$

Separating the BCs gives

$$\begin{aligned} u(0,y,t) &= 0 \quad \Rightarrow \quad X(0) = 0 \\ u(L,y,t) &= 0 \quad \Rightarrow \quad X(L) = 0 \\ \frac{\partial u}{\partial y}(x,0,t) &= 0 \quad \Rightarrow \quad Y'(0) = 0 \\ \frac{\partial u}{\partial y}(x,H,t) &= 0 \quad \Rightarrow \quad Y'(L) = 0 \end{aligned}$$

Rearranging the other part of the equation gives

$$k_2 \frac{Y''(y)}{Y(y)} + \lambda = -k_1 \frac{X''(x)}{X(x)}$$

Since the l.h.s. depends on y and the r.h.s. on x, both sides must equal a constant, say μ,

$$k_2 \frac{Y''(y)}{Y(y)} + \lambda = -k_1 \frac{X''(x)}{X(x)} = \mu$$

The problem for $X(x)$ is now

$$X''(x) + \frac{\mu}{k_1} X(x) = 0; \qquad X(0) = 0 = X(L)$$

The solution is, as we found in class,

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \mu_n = k_1 \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3...$$

The problem for $Y(y)$ is

$$Y''(y) + \frac{\lambda - \mu_n}{k_2} Y(y) = 0; \qquad Y'(0) = 0 = Y'(H).$$

We've found the solution to this problem before,

$$Y_m(y) = \cos\left(\frac{m\pi y}{H}\right), \qquad \frac{\lambda_{mn} - \mu_n}{k_2} = \left(\frac{m\pi}{H}\right)^2, \qquad m = 1, 2, 3...$$

Thus

$$\lambda_{mn} = k_2 \left(\frac{m\pi}{H}\right)^2 + \mu_n = \pi^2 \left(k_2 \frac{m^2}{H^2} + k_1 \frac{n^2}{L^2}\right), \qquad m, n = 1, 2, 3... \tag{7}$$

and the solutions to the PDE and BCs are

$$u_{mn}(x, y, t) = \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) e^{-\lambda_{mn} t}, \qquad m, n = 1, 2, 3...$$

Using superposition, we sum over all m, n to obtain the general solution,

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} u_{mn}(x, y, t) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) e^{-\lambda_{mn} t} \end{aligned}$$

where the constants A_{mn} are found by imposing the IC:

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right)$$

Using the orthogonality of sines and cosines, we multiply by $\sin(k\pi x/L)\cos(l\pi y/H)$ and integrate in x from 0 to L, and in y from 0 to H, to obtain

$$\begin{aligned}
& \int_{x=0}^{L}\int_{y=0}^{H} f(x,y)\sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{l\pi y}{H}\right)dydx \\
= & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\int_{x=0}^{L}\int_{y=0}^{H}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi y}{H}\right)dydx \\
= & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\int_{x=0}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx\int_{y=0}^{H}\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi y}{H}\right)dy \\
= & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{L}{2}\delta_{nk}\frac{H}{2}\delta_{ml} \\
= & A_{lk}\frac{LH}{4}
\end{aligned}$$

Hence

$$A_{mn} = \frac{4}{LH}\int_{x=0}^{L}\int_{y=0}^{H} f(x,y)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dydx$$

3 Problem 3

Haberman Problem 7.7.4 (a), p. 316. The pie-shaped membrane problem.

Solution: We consider the displacement on a pie shaped membrane of radius a and angle $\pi/3$ radians that satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

Due to the circular geometry, we use polar coordinates and separate variables as

$$u = R(r)\,H(\theta)\,T(t)$$

Recall that in polar coords the Laplacian is given by

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

and hence the PDE becomes

$$\frac{T''(t)}{c^2 T(t)} = \frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{r^2 H(\theta)}\frac{d^2 H(\theta)}{d\theta^2}$$

The l.h.s. depends on t and the r.h.s. on (r,θ), so that both sides equal a constant $-\lambda$ (the separation constant)

$$\frac{T''(t)}{c^2 T(t)} = \frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{r^2 H(\theta)}\frac{d^2 H(\theta)}{d\theta^2} = -\lambda$$

Recall that we showed in class that if the BCs are Type I or Type II then the eigenvalues are positive. In all the cases in the problem, the boundary either has Type I or Type II BCs, so that $\lambda > 0$. Thus

$$T(t) = a\cos\left(c\sqrt{\lambda}t\right) + b\sin\left(c\sqrt{\lambda}t\right)$$

The BCs are

$$\begin{aligned} u(r,0,t) &= 0 &\Rightarrow\quad & H(0)=0, \\ u(r,\pi/3,t) &= 0 &\Rightarrow\quad & H(\pi/3)=0, \\ \frac{\partial u}{\partial r}(a,\theta,t) &= 0 &\Rightarrow\quad & R'(a)=0. \end{aligned}$$

Since we have a complete set of homogeneous BCs for $H(\theta)$, we solve for H first. Notice that going from (x,y) to (r,θ), the sides of the membrane don't touch, unlike the disc, and hence $H(\theta)$ does not need to be periodic.

The equation for $R(r)$ and $H(\theta)$ can be rearranged:

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \lambda r^2 = -\frac{1}{H(\theta)}\frac{d^2H(\theta)}{d\theta^2}$$

Since the l.h.s. depends on r and the r.h.s. on θ, both sides must equal a constant μ:

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \lambda r^2 = -\frac{1}{H(\theta)}\frac{d^2H(\theta)}{d\theta^2} = \mu$$

Thus the problem for $H(\theta)$ is

$$\frac{d^2H(\theta)}{d\theta^2} + \mu H(\theta) = 0; \qquad H(0) = 0 = H(\pi/3).$$

This is just the 1D Sturm Liouville problem with Type I homogeneous BCs. We have shown before that $\mu > 0$ and the eigenfunctions and eigenvalues are

$$H_n(\theta) = \sin\left(\frac{n\pi\theta}{\pi/3}\right) = \sin(3n\theta), \qquad \mu_n = \left(\frac{n\pi}{\pi/3}\right)^2 = 9n^2, \qquad n = 1,2,3...$$

The equation for R is then, on rearranging,

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left(\left(\sqrt{\lambda}r\right)^2 - (3n)^2\right)R = 0$$

This is the Bessel Equation of order $3n$. The general solution is

$$R_n(r) = a_n J_{3n}\left(\sqrt{\lambda}r\right) + b_n Y_{3n}\left(\sqrt{\lambda}r\right), \qquad n = 1,2,3...$$

We must impose that the temperature at $r = 0$ (the origin) remains bounded, and hence must take $b_n = 0$. Thus

$$R_n(r) = a_n J_{3n}\left(\sqrt{\lambda} r\right), \qquad n = 1, 2, 3...$$

Imposing the BC at $r = a$ gives

$$0 = R_n'(a) = a_n \sqrt{\lambda} J_{3n}'\left(\sqrt{\lambda} a\right)$$

Thus the eigenvalues are given by solving

$$J_{3n}'\left(\sqrt{\lambda_{nm}} a\right) = 0, \qquad n, m = 1, 2, 3...$$

Just like the zeros of the Bessel function, there are infinitely many local max and mins, and hence J_{3n}' has infinitely many zero. We enumerate the zeros with the index m, and the order of the Bessel function with the index n.

The solution to the PDE and BCs is thus

$$u_{nm} = J_{3n}\left(\sqrt{\lambda_{nm}} r\right) \sin(3n\theta)\left(a_{nm}\cos\left(c\sqrt{\lambda_{nm}} t\right) + b_{nm}\sin\left(c\sqrt{\lambda_{nm}} t\right)\right), \qquad n, m = 1, 2, 3...$$

Using superposition, we sum over all n, m to obtain the general solution

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_{3n}\left(\sqrt{\lambda_{nm}} r\right) \sin(3n\theta)\left(a_{nm}\cos\left(c\sqrt{\lambda_{nm}} t\right) + b_{nm}\sin\left(c\sqrt{\lambda_{nm}} t\right)\right)$$

where the a_{nm} and b_{nm} are found from the ICs. The natural frequencies of oscillation are

$$f_{nm} = \frac{c\sqrt{\lambda_{nm}}}{2\pi}, \qquad n, m = 1, 2, 3...$$

4 Problem 4

Find the eigenvalue λ and corresponding eigenfunction v for the 30^o-60^o-90^o right triangle (i.e. a right triangle that has these angles); v and λ satisfy

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where $D = \left\{(x, y) : 0 < y < \sqrt{3}x, \quad 0 < x < 1\right\}$.

Hint: combine the eigenfunctions on the rectangle

$$D = \left\{(x, y) : 0 < x < 1, 0 < y < \sqrt{3}\right\}$$

to obtain an eigenfunction on D that is positive on D. We know that the first eigenfunction can be characterized (up to a non-zero multiplicative constant) as the eigenfunction that is of one sign. You may use the eigenfunctions derived in-class for the rectangle, without derivation. Be sure to sketch the region correctly before solving the problem.

Solution: Continuing where the notes left off, the trial function is

$$v_T = Av_{31} + Bv_{24} + Cv_{15}$$

where

$$v_{mn} = \sin(m\pi x)\sin\left(n\pi\frac{y}{\sqrt{3}}\right)$$

and A, B, C are constants determined to make $v_T = 0$ on $y = \sqrt{3}x$,

$$Av_{31} + Bv_{24} + Cv_{15} = 0, \qquad y = \sqrt{3}x.$$

Note that $v_T = 0$ on $x = 1$ and $y = 0$ since v_{31}, v_{24}, v_{15} are eigenfunctons for the rectangle $\left\{0 < x < 1, 0 < y < \sqrt{3}\right\}$. We will use the fact that

$$\begin{aligned}
\sin a \sin b &= \frac{1}{2}(\cos(a-b) - \cos(a+b)) \\
\cos(-a) &= \cos(a)
\end{aligned}$$

Thus, on $y = \sqrt{3}x$,

$$\begin{aligned}
0 &= Av_{31} + Bv_{24} + Cv_{15} \\
&= A\sin(3\pi x)\sin(\pi x) + B\sin(2\pi x)\sin(4\pi x) + C\sin(\pi x)\sin(5\pi x) \\
&= \frac{A}{2}(\cos(2\pi x) - \cos(4\pi x)) + \frac{B}{2}(\cos(-2\pi x) - \cos(6\pi x)) \\
&\quad + \frac{C}{2}(\cos(-4\pi x) - \cos(6\pi x)) \\
&= \frac{1}{2}(A+B)\cos(2\pi x) + \frac{1}{2}(-A+C)\cos(4\pi x) + \frac{1}{2}(-B-C)
\end{aligned}$$

Since $\cos(2\pi x)$, $\cos(4\pi x)$, $\cos(6\pi x)$ are linearly independent, each coefficient must be zero:

$$A + B = 0, \qquad -A + C = 0, \qquad -B - C = 0$$

There are only two independent equations in these 3 (i.e you can get any one of the equations by combining the other two). Thus we are left with a multiplicative constant C, which is fine because eigen-functions are only defined up to a multiplicative constant (we often just take the constant to be 1, for simplicity). In summary, the function

$$v_T = Av_{31} + Bv_{24} + Cv_{15} = C(v_{31} - v_{24} + v_{15})$$

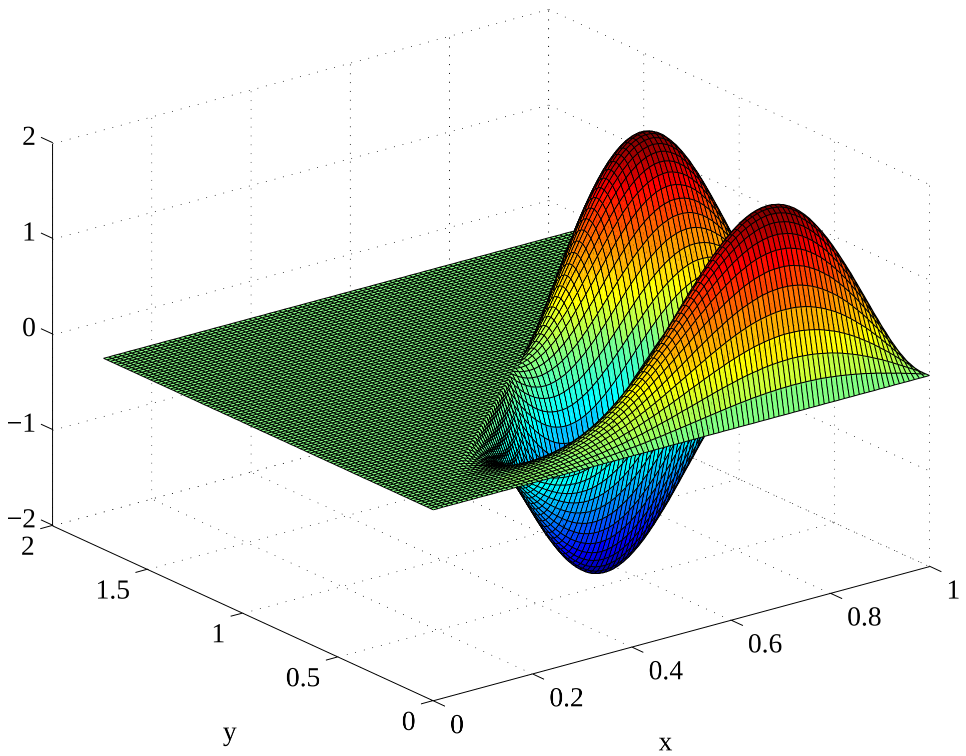

Figure 2: Plot of $v_T(x, y)$.

is zero on the boundary of D_T. Also, since each of v_{31}, v_{24}, v_{15} has eigenvalue $\lambda_{31} = \lambda_{24} = \lambda_{15} = 28\pi^2/3$, we have

$$\nabla^2 v_T = C\left(\nabla^2 v_{31} - \nabla^2 v_{24} + \nabla^2 v_{15}\right) = C\frac{28}{3}\pi^2\left(v_{31} - v_{24} + v_{15}\right) = \frac{28}{3}\pi^2 v_T$$

in the triangle. Thus v_T is an eigen-function on the triangle with eigenvalue $28\pi^2/3$.

[extra] Actually v_T does equal zero on the interior (see Figure 2). So other methods would have to be used to find the smallest eigen-value.

We can get an upper bound on v by using the Rayleigh quotient, and associated theorems. Consider the function

$$v = y\,(x-1)\left(y - \sqrt{3}x\right)$$

Note that $v = 0$ on the boundary ($y = 0$, $x = 1$, $y = \sqrt{3}x$), v is nonzero on the interior, and

$$\begin{aligned} |\nabla v|^2 &= \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \\ &= y^2\left(y - 2\sqrt{3}x + \sqrt{3}\right)^2 + (x-1)^2\left(2y - \sqrt{3}x\right)^2 \end{aligned}$$

The Rayleigh quotient is

$$\begin{aligned} R(v) &= \frac{\int_0^1 \left(\int_0^{\sqrt{3}x} \left(y^2\left(y-2\sqrt{3}x+\sqrt{3}\right)^2+(x-1)^2\left(2y-\sqrt{3}x\right)^2\right)dy\right)dx}{\int_0^1 \left(\int_0^{\sqrt{3}x} y^2 (x-1)^2 \left(y-\sqrt{3}x\right)^2 dy\right)dx} \\ &= \frac{\frac{1}{15}\sqrt{3}}{\frac{1}{560}\sqrt{3}} = \frac{112}{3} \approx 37.3 \end{aligned}$$

Thus by our theorem, the smallest eigenvalue on the triangle is less than 37.3. In particular, the eigenvalue associated with v_T was

$$\lambda_T = \frac{28}{3}\pi^2 \approx 92.1$$

Clearly, this is NOT the smallest eigenvalue.

5 Problem 5

Consider the boundary value problem on the isosceles right angled triangle of side length 1,

$$\nabla^2 v = 0, \qquad 0 < y < x, \qquad 0 < x < 1$$

subject to the BCs

$$\begin{aligned} \frac{\partial v}{\partial x}(1,y) &= 0, \qquad 0 < y < 1 \\ \frac{\partial v}{\partial y}(x,0) &= 0, \qquad 0 < x < 1 \\ v(x,x) &= 0, \qquad 0 < x < 1/2 \\ v(x,x) &= 50, \qquad 1/2 < x < 1 \end{aligned}$$

Give a symmetry argument to show that $v(x, 1-x) = 25$ for $0 < x < 1$. Sketch the level curves and heat flow lines of v. Be sure to sketch the region correctly before solving the problem.

Solution: Take any point $P = (x_0, 1-x_0)$ on the line $y = 1-x$ inside the triangle. Let $v_1 = v(x_0, 1-x_0)$ be its steady-state temperature. Rotate the triangle about the line $y = 1-x$. When you do this, the position of the point P doesn't change (see Figure 3(top) for a schematic of this argument). Add the solutions on the two plates to obtain a triangular plate insulated on the horizontal and vertical side, and kept at $u = 50$ on the hypotenuse. Thus the steady state temperature of this new plate is 50. Hence $v_1 + v_1 = 50$, or $v_1 = 25$. Since the point we chose was arbitrary, then all points on the line $y = 1-x$ have steady state temp $v(x, 1-x) = 25$ for

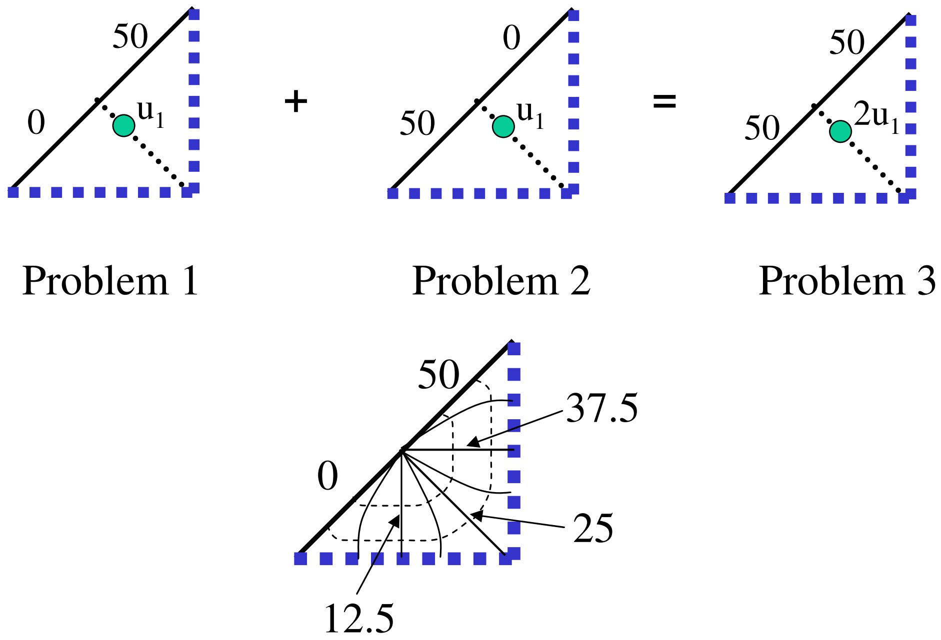

Figure 3: Top: schematic of symmetry argument. Bottom: sketch of level curves (solid) and heat flow lines (broken) of $u(x,y)$.

$0 < x < 1$ (and inside the triangle, or $1/2 < x < 1$). The level curves and heat flow lines are given as solid and dashed lines, respectively, in Figure 3(bottom). [extra] Note also that $v\,(x, 0.5) = 75/2$ for $1/2 < x < 1$ and $v\,(0.5, y) = 25/2$ for $0 < y < 1/2$.