Fourier Sine Series Examples

March 16, 2009

The Fourier sine series for a function $f(x)$ defined on $x \in [0,1]$ writes $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

for some coefficients b_n. The key point is that these functions are *orthogonal*, given the "dot product" $f(x) \cdot g(x) = \int_0^1 f(x)g(x)dx$. It is a simple calculus exercise to show that the dot product of two sine functions is $\sin(n\pi x) \cdot \sin(m\pi x) = \int_0^1 \sin(n\pi x)\sin(m\pi x) = \int_0^1 [\cos((n-m)\pi x) - \cos((n+m)\pi x)]/2$, which equals 0 if $n \neq m$ and $1/2$ if $n = m$. [If we divide the $\sin(n\pi x)$ functions by $\sqrt{1/2}$, they are *orthonormal.*] Because of orthogonality, we can compute the b_n very simply: for any given m, we integrate both sides against $\sin(m\pi x)$. In the summation, this gives zero for $n \neq m$, and $\int_0^1 \sin^2(m\pi x) = 1/2$ for $n = m$, resulting in the equation

$$b_m = 2\int_0^1 f(x)\,\sin(m\pi x)\,dx.$$

Fourier claimed (without proof) in 1822 that *any* function $f(x)$ can be expanded in terms of sines in this way, even discontinuous function! That is, these sine functions form an *orthogonal basis* for "all" functions! This turned out to be false for various badly behaved $f(x)$, and controversy over the exact conditions for convergence of the Fourier series lasted for well over a century, until the question was finally settled by Carleson (1966) and Hunt (1968): any function $f(x)$ where $\int |f(x)|^p dx$ is finite for some $p > 1$ has a Fourier series that converges *almost everywhere* to $f(x)$ [except at isolated points]. At points where $f(x)$ has a jump discontinuity, the Fourier series converges to the midpoint of the jump. So, as long as one does not care about crazy divergent functions or the function value exactly at points of discontinuity (which usually has no physical significance), Fourier's remarkable claim is essentially true.

To illustrate the convergence of the sine series, let's consider a couple of examples. First, consider the function $f(x) = 1$, which seems impossible to expand in sines because it is not zero at the endpoints, but nevertheless it works...if you don't care about the value *exactly* at $x = 0$ or $x = 1$. From the formula above, we obtain

$$b_m = 2\int_0^1 \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^1 = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases},$$

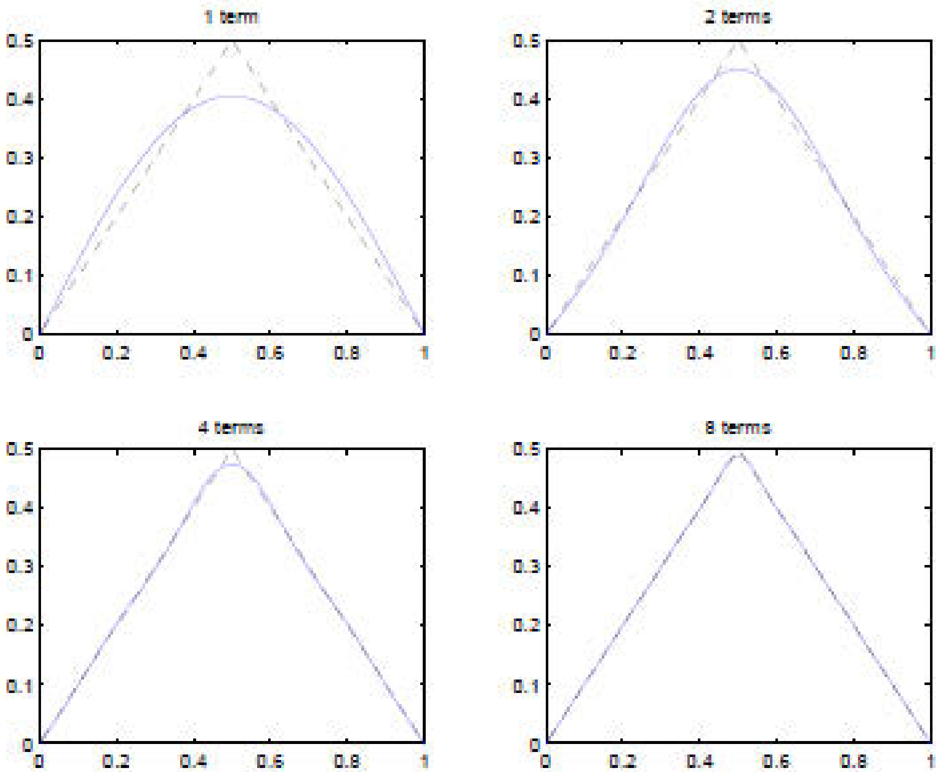

Figure 2: Fourier sine series (blue lines) for the triangle function $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ (dashed black lines), truncated to a finite number of terms (from 1 to 32), showing that the series indeed converges everywhere to $f(x)$.

the region:

$$b_m \text{ odd} = 2\int_0^1 f(x)\sin(m\pi x)\,dx = 4\int_0^{1/2} x\sin(m\pi x)\,dx = \frac{4}{(m\pi)^2}(-1)^{\frac{m-1}{2}},$$

where for the last step one must do some tedious integration by parts, and thus

$$f(x) = \frac{4}{\pi^2}\sin(\pi x) - \frac{4}{(3\pi)^2}\sin(3\pi x) + \frac{4}{(5\pi)^2}\sin(5\pi x) + \cdots.$$

This is plotted in figure. 2 for 1 to 8 terms—it converges faster than for $f(x) = 1$ because there are no discontinuities in the function to match, only discontinuities in the derivative.

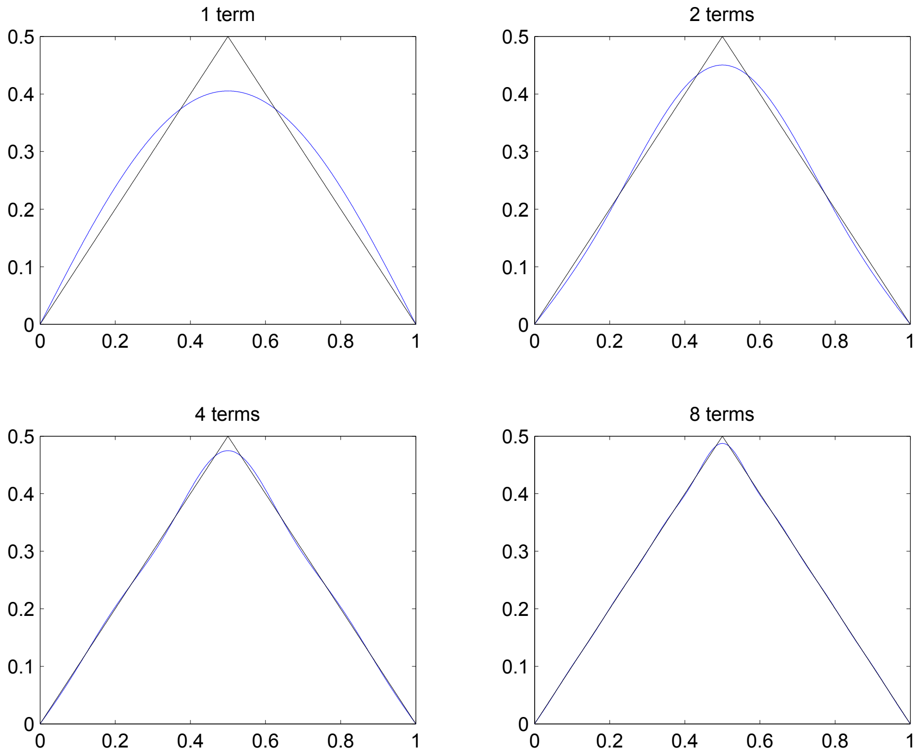

Figure 2: Fourier sine series (blue lines) for the triangle function $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ (dashed black lines), truncated to a finite number of terms (from 1 to 32), showing that the series indeed converges everywhere to $f(x)$.

the region:

$$b_{m \text{ odd}} = 2\int_0^1 f(x)\,\sin(m\pi x)\,dx = 4\int_0^{1/2} x\sin(m\pi x)\,dx = \frac{4}{(m\pi)^2}(-1)^{\frac{m-1}{2}},$$

where for the last step one must do some tedious integration by parts, and thus

$$f(x) = \frac{4}{\pi^2}\sin(\pi x) - \frac{4}{(3\pi)^2}\sin(3\pi x) + \frac{4}{(5\pi)^2}\sin(5\pi x) + \cdots.$$

This is plotted in figure. 2 for 1 to 8 terms—it converges faster than for $f(x) = 1$ because there are no discontinuities in the function to match, only discontinuities in the derivative.