18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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The first problem set is posted, and it is due March 4th. There are 15 problems, but we only have to do about 6 of them. Later on, we will have one lecture to discuss the problem set solutions.

Let's talk more about q-binomial coefficients. We have the normal binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Unfortunately, q-binomial coefficients are less nice, especially since in a "quantum world," x and y may not always commute. Define

$$yx = qxy \text{ such that } qx = xq, qy = yq$$

Then we have the following:

Proposition 1

For any x, y, n,

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

For example, taking $n = 2$,

$$(x+y)^2 = (x+y)(x+y) = x^2 + yx + xy + y^2 = x^2 + (1+q)xy + y^2$$

In the previous lecture, we proved a main theorem:

Theorem 2

Defining

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where $[n]_= 1 + q + \cdots + q^{n-1}$ and factorials multiply q-numbers, then we have the identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

Somehow this identity still looks like a miracle. Where's the actual relation? Let's find a more conceptual proof!

Fact 3

We're going to use linear algebra, since this class is called *algebraic combinatorics.*

Definition 4

Define the **Grassmannian** Gr_{kn} to be the space of k-dimensional linear subspaces of n-dimensional space.

Let's think about this more concretely: what's the hands-down approach? The idea is that elements of the Grassmannian are defined by matrices. We pick a linearly independent basis and put those vectors down as rows!

Proposition 5

We can treat Gr_{kn} as the set of $k \times n$ matrices of rank k. Then if we perform row operations, we have identical spaces, so we want to use the matrices **modulo row operations.**

So how can we count these? We haven't mentioned what the entries of the matrices are! They could be real or complex, or we could pick any finite field as well.

Fact 6

For any $q = p^r$ for p prime and r natural number, there exists a unique finite field of order q up to isomorphism. It is denoted $\mathbb{F}_q$.

This is closely related to many fields! Now, we can rephrase our question better:

Question 7. What is the number of entries in $\mathrm{Gr}_{kn}(\mathbb{F}_q)$?

We're going to count this in two different ways.

Solution 1. First of all, note that row operations are $k \times k$ invertible matrices, so we can calculate the number of $k \times n$ matrices of rank k, **divided by** the number of $k \times k$ invertible matrices!

A matrix A has rows $v_1, v_2, \cdots, v_k$, where v_i are all vectors in $\mathbb{F}_q^n$. To make sure A is invertible, all v_k must be independent. The number of ways to pick v_1 is $q^n - 1$ (it can be anything except the zero vector). Next, v_2 can be anything except multiples of v_1, so there are $q^n - q$ ways to pick v_2. Similarly, there are $q^n - q^2$ ways to pick v_3, and so on, so the total number of $k \times n$ matrices of rank k is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}).$$

But we need to divide by the number of $k \times k$ invertible matrices, so our total expression is just

$$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$

With a tiny bit of simplification, this is exactly the definition of the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$! It's only for q prime powers though. □

Here's another way to use a notion from linear algebra!

Solution 2. We can transform A into **reduced row-echelon form.** The idea is to use Gaussian elimination.

Fact 8

This was developed by Newton in 1670, then also Gauss in 1810, and also Jordan in 1888), but also it was known to Chinese mathematicians in 179 AD or even 150 BC. Basically, ancient mathematicians already knew about this.

Here's the general idea: find the first nonzero column. Make the top left corner 1 with row operations if at least one entry in the first column is 1. This can kill all entries below it. Move on to the next column, and either kill everything in the lower columns or start a new pivot.

For example, for $k = 5, n = 10$, an example of a reduced form matrix is

$$\begin{bmatrix} 1 & * & 0 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 1 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

The 1s are called **pivots.** The pivot columns are pretty boring, so we can just remove them:

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Fact 9

Does this remind us of anything? One student once said an American flag, but the idea is that it looks like a Young diagram!

So how many such shapes are there? It's the number of tableaux that fit in an $k \times (n-k)$ rectangle! So since the $*$s can be anything in our matrices, we again have

Theorem 10

The number of elements in the Grassmannian $\mathrm{Gr}_{kn}(\mathbb{F}_q)$ is

$$\sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

$\square$

So we've showed the identity that we want for any q is a power of a prime. To finish, we go back to Euclid!

Fact 11 (300 BC, Euclid)

There are infinitely many prime numbers.

(Proof: assume finitely many, multiply all together and add 1. This new number is not divisible by any primes, contradiction.)

We also use a fact from algebra:

Fact 12

If two rational expressions with integer coefficients $f(q)$ and $g(q)$ are identical on infinitely many values, they are the same on all values of q!

This means that we have our result:

Corollary 13

For all values of n, k, q,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

What's left is unimodality, but we'll do that later. Here's a fun fact:

Definition 14

Given a permutation $w = (w_1, \cdots, w_n) \in S_n$, (i, j) is an **inversion** of w if $i < j$ and $w_i > w_j$. Denote $\text{inv}(w)$ to be the number of inversions of w.

Example 15

$\text{inv}(312675)$ is 4: $31, 32, 65, 75$.

Theorem 16

We have

$$[n]_q! = \sum_{w \in S_n} q^{\text{inv}(w)}.$$

So the size of the Young diagram plays the same role as the number of inversions! We'll see that there's something more general going on in the future.

Proof. We will prove this by induction. How do we grow a permutation? Start with $u \in S_{n-1}$, and then we can add n in n different places. If it goes at the end, it adds no inversions: if we add it second-to-last, we get 1 inversion, and so on. So this multiplies on a factor of $(1 + q + \cdots + q^{n-1})$, which is exactly what we want: the q-number for n. $\square$