18.175: Lecture 38

Even more Brownian motion

Scott Sheffield

MIT

Outline

Recollections

Markov property, Blumenthal's 0-1 law

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Basic properties

- Brownian motion is real-valued process B_t, $t \geq 0$.
- **Independent increments:** If $t_0 < t_1 < t_2 \ldots$ then $B(t_0), B(t_1 - t_0), B(t_2 - t_1), \ldots$ are independent.
- **Gaussian increments:** If $s, t \geq 0$ then $B(s + t) - B(s)$ is normal with variance t.
- **Continuity:** With probability one, $t \to B_t$ is continuous.
- Hmm... does this mean we need to use a σ-algebra in which the event "B_t is continuous" is a measurable?
- Suppose Ω is set of all functions of t, and we use smallest σ-field that makes each B_t a measurable random variable... does that fail?

Basic properties

- Translation invariance: is $B_{t_0+t} - B_{t_0}$ a Brownian motion?
- Brownian scaling: fix c, then B_{ct} agrees in law with $c^{1/2}B_t$.
- Another characterization: B is jointly Gaussian, $EB_s = 0$, $EB_sB_t = s \wedge t$, and $t \to B_t$ a.s. continuous.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.
- But not enough to get a.s. continuity.
- Can define Brownian motion jointly on diadic rationals pretty easily. And claim that this a.s. extends to continuous path in unique way.
- We can use the Kolmogorov continuity theorem (next slide).
- Can prove Hölder continuity using similar estimates (see problem set).
- Can extend to higher dimensions: make each coordinate independent Brownian motion.

Continuity theorem

- **Kolmogorov continuity theorem:** Suppose $E|X_s - X_t|^\beta \le K|t-s|^{1+\alpha}$ where $\alpha, \beta > 0$. If $\gamma < \alpha/\beta$ then with probability one there is a constant $C(\omega)$ so that $|X(q) - X(r)| \le C|q-r|^\gamma$ for all $q, r \in \mathbb{Q}_2 \cap [0,1]$.
- **Proof idea:** First look at values at all multiples of 2^{-0}, then at all multiples of 2^{-1}, then multiples of 2^{-2}, etc.
- At each stage we can draw a nice piecewise linear approximation of the process. How much does the approximation change in supremum norm (or some other Hölder norm) on the ith step? Can we say it probably doesn't change very much? Can we say the sequence of approximations is a.s. Cauchy in the appropriate normed spaced?

Continuity theorem proof

- **Kolmogorov continuity theorem:** Suppose $E|X_s - X_t|^\beta \leq K|t-s|^{1+\alpha}$ where $\alpha, \beta > 0$. If $\gamma < \alpha/\beta$ then with probability one there is a constant $C(\omega)$ so that $|X(q) - X(r)| \leq C|q-r|^\gamma$ for all $q, r \in \mathbb{Q}_2 \cap [0,1]$.
- **Argument from Durrett (Pemantle):** Write

$$G_n = \{|X(i/2^n) - X((i-1)/2^n)|\} \leq C|q-r|^\lambda \,\mathrm{for}\, 0 < i \leq 2^n\}.$$

- Chebyshev implies $P(|Y| > a) \leq a^{-\beta} E|Y|^\beta$, so if $\lambda = \alpha - \beta\gamma > 0$ then

$$P(G_n^c) \leq 2^n \cdot 2^{n\beta\gamma} \cdot E|X(j2^{-n})|^\beta = K2^{-n\lambda}.$$

Easy observations

- Brownian motion is Hölder continuous for any $\gamma < 1/2$ (apply theorem with $\beta = 2m, \alpha = m - 1$).
- Brownian motion is almost surely not differentiable.
- Brownian motion is almost surely not Lipschitz.
- Kolmogorov-Centsov theorem applies to higher dimensions (with adjusted exponents). One can construct a.s. continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

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More σ-algebra thoughts

- Write $\mathcal{F}_s^o = \sigma(B_r : r \leq s)$.
- Write $\mathcal{F}_s^+ = \cap_{t>s}\mathcal{F}_t^o$
- Note right continuity: $\cap_{t>s}\mathcal{F}_t^+ = \mathcal{F}_s^+$.
- $\mathcal{F}_s^+$ allows an "infinitesimal peek at future"

Markov property

- If $s \geq 0$ and Y is bounded and $\mathcal{C}$-measurable, then for all $x \in \mathbb{R}^d$, we have
$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y,$$
where the RHS is function $\phi(x) = E_x Y$ evaluated at $x = B_s$.
- **Proof idea:** First establish this for some simple functions Y (depending on finitely many time values) and then use measure theory (monotone class theorem) to extend to general case.

Looking ahead

- **Expectation equivalence theorem** If Z is bounded and measurable then for all $s \geq 0$ and $x \in \mathbb{R}^d$ have

$$E_x(Z|\mathcal{F}_s^+) = E_x(Z|\mathcal{F}_s^o).$$

- **Proof idea:** Consider case that $Z = \sum_{i=1}^m f_m(B(t_m))$ and the f_m are bounded and measurable. Kind of obvious in this case. Then use same measure theory as in Markov property proof to extend general Z.
- **Observe:** If $Z \in \mathcal{F}_s^+$ then $Z = E_x(Z|\mathcal{F}_s^o)$. Conclude that $\mathcal{F}_s^+$ and $\mathcal{F}_s^o$ agree up to null sets.

Blumenthal's 0-1 law

- If $A \in \mathcal{F}_0^+$, then $P(A) \in \{0,1\}$ (if P is probability law for Brownian motion started at fixed value x at time 0).
- There's nothing you can learn from infinitesimal neighborhood of future.
- **Proof:** If we have $A \in \mathcal{F}_0^+$, then previous theorem implies

$$1_A = E_x(1_A|\mathcal{F}_0^+) = E_x(1_A|\mathcal{F}_0^o) = P_x(A) \quad P_x\text{a.s.}$$

More observations

- If $\tau = \inf\{t \geq 0 : B_t > 0\}$ then $P_0(\tau = 0) = 1$.
- If $T_0 = \inf\{t > 0 : B_t = 0\}$ then $P_0(T_0 = 0) = 1$.
- If B_t is Brownian motion started at 0, then so is process defined by $X_0 = 0$ and $X_t = tB(1/t)$. (Proved by checking $E(X_s X_t) = stE(B(1/s)B(1/t)) = s$ when $s < t$. Then check continuity at zero.)

Continuous martingales

- What can we say about continuous martingales?
- Do they all kind of look like Brownian motion?